

НАЦІОНАЛЬНИЙ УНІВЕРСИТЕТ «ОДЕСЬКА ПОЛІТЕХНІКА»
МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ

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SCUOLA DI DOTTORATO

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Кваліфікаційна наукова
праця на правах рукопису

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ДИСЕРТАЦІЯ

**PERTURBATIVE AND NON-PERTURBATIVE ASPECTS OF
HADRONIC PHYSICS**

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UNIVERSITÀ DEGLI STUDI DI
TORINO



ODESA POLYTECHNIC NATIONAL
UNIVERSITY

**PERTURBATIVE AND NON-PERTURBATIVE ASPECTS OF
HADRONIC PHYSICS**

Tetiana Yushkevych
September 2025

*A dissertation proposal in partial fulfillment for the degree of
Doctor of Philosophy in Physics*

Анотація

Юшкевич Т.В. Пертурбативні та непертурбативні аспекти адронної фізики — Кваліфікаційна наукова праця на правах рукопису.

Дисертація на здобуття наукового ступеня кандидата фізико-математичних наук (доктора філософії) за спеціальністю 104 Фізика та астрономія (10 — Природничі науки). — Національний університет «Одеська політехніка» МОН України, Одеса, та Туринський Університет, Турин (Італія), 2025.

Ця дисертація представляє комбіноване теоретичне та феноменологічне дослідження динаміки квантової хромодинаміки (КХД), зосереджуючись на двох взаємодоповнюючих аспектах.

По-перше, метод багаточастинкових полів використовується для побудови моделей процесів розсіювання протонів (антипротонів). Для апроксимації внесків петлевих діаграм у рамках цієї системи використовується метод Лапласа, який пропонує практичний, але потребуючий вдосконалення обчислювальний інструмент. Ця робота демонструє, що метод багаточастинкових полів дозволяє будувати динамічну модель для опису процесів розсіювання протонів як системи кварків, досягаючи якісної узгодженості з експериментальними спостереженнями. Зокрема, застосування методу Лапласа забезпечує наближений, хоча й обчислювально вимогливий, підхід

до оцінки петлевих діаграм у цих рамках. Показано, що експериментально спостережувана немонотонна поведінка диференціального перерізу пружного розсіювання протонів як функції квадрата переданого імпульсу виникає через спінові ефекти, які, будучи включеними як у деревоподібні, так і в прості петлеві діаграми, призводять до якісної узгодженості з експериментальними даними.

По-друге, розроблено методологію на основі спорідненості для дослідження кінематичного переходу між режимами поперечної залежності від імпульсу (TMD) та колінеарної факторизації в напівінклюзивному глибоконепружному розсіюванні (SIDIS). З цією метою пропонується детальна кінематична оцінка фазового простору SIDIS з використанням підходу на основі інструменту "афініті". Ця методологія дозволяє надійно ідентифікувати області з домінуванням TMD, колінеарних теорем факторизації, а також області де вони збігаються, що надає нові можливості для дослідження механізмів факторизації та процесів адронізації. Перехід від усереднених по бінах до розрахунків "афініті" для кожної події зберігає суттєві кореляції між непертурбативними параметрами, що призводить до більш надійної класифікації кінематичних областей. Аналіз показує, що область TMD є ширшою, ніж очікувалося раніше, особливо в межах кінематики JLab12, тоді як колінеарна область стає добре доступною лише при вищих енергіях, таких як у JLab22 та електронно-іонного колайдера. У рамках використання "афініті" стала можливою ідентифікація окремих TMD, колінеарної та TMD-колінеарного узгодження областей, що відкриває нові перспективи

для вивчення взаємодії між режимами факторизації.

Результати цієї дисертації пропонують як теоретичні висновки, так і практичні інструменти для майбутніх досліджень структури нуклонів та процесів пружного розсіювання. Рекомендації для подальшого розвитку включають включення петльових діаграм вищого порядку в структуру розрахунків за методом багаточастинкових полів, вдосконалення обчислювальних методів, розширення аналізів на основі "афініті" на інші процеси та спрямування експериментальних зусиль через цільові кінематичні розрізи.

Ключові слова: Фізика високих енергій: феноменологія та експеримент; Квантова хромодинаміка (КХД): партони, кварки та глюони; Процеси жорсткого розсіювання; Багатокваркове розсіювання; Пружне розсіювання протонів; Спінова фізика; Апроксимація петльових діаграм; Інклюзивне та напівінклюзивне глибоконепружне розсіювання (DIS та SIDIS); Структура нуклона; Поперечна залежність від імпульсу (TMD); Колінеарна та TMD факторизація; Феноменологічне та математичне моделювання; Методи Монте-Карло; Нейронні мережі; Ядерні реакції

Список публікацій здобувача:

Основні результати дисертаційної роботи викладено в 4 публікаціях, 3 з яких індексуються в наукометричній базі Scopus (2 статті опубліковано у виданні, віднесеному до першого квартилю (Q1)

відповідно до класифікації SCImago Journal and Country Rank). Додатково результати висвітлено у 5 тезах доповідей на міжнародних конференціях.

- *журнальні публікації:*

1. Chudak N., Sharph I., Potiienko O., Milieva A., Merkotan K., Yushkevych T. and Rusov V., Laplace's method for elastic scattering diagrams within multiparticle fields model, Phys. Review D 106, 3 (2022) DOI:<https://doi.org/10.1103/PhysRevD.106.036023>
2. O. Potiienko, K. Merkotan, N. Chudak, D. Ptashynskiy, T. Zelentsova, T. Yushkevich, I. Sharph, and V. Rusov, New method of accounting for interference contributions within a multiperipheral model, Physical Review D 101, 7 (2020) DOI:<https://doi.org/10.1103/PhysRevD.101.076021>
3. D.A. Ptashynskiy, T.M. Zelentsova, N.O. Chudak, K.K. Merkotan, O.S. Potiienko, V.V. Voitenko, O.D. Berezovskiy, V.V. Opyatyuk, O.V. Zhavorova, T.V. Yushkevich, I.V. Sharph, V.D. Rusov, Multiparticle fields on the subset of simultaneity, Ukrainian Journal of Physics 64(8), 732 (2019). DOI: <https://doi.org/10.15407/ujpe64.8.732>

Апробація результатів дисертації

- *тези конференцій:*

4. T. Yushkevych, M. Boglione, A. Prokudin, "Affinity tool. Comparison between JLab12 and JLab22", "Science at the Luminosity

Frontier: Jefferson Lab at 22 GeV” (Frascati, Italy, 2024) (<https://agenda.infn.it/event/39742/contributions/247698/>)

5. T. Yushkevych, I. Sharph, et.al, “Laplace method for single-loop diagrams of elastic proton scattering”, 21-th Gamow International Astronomical Conference-School “Astronomy and beyond: astrophysics, cosmology and gravitation, high energy physics, astroparticle physics, radioastronomy and astrobiology”, August 15-21, 2021 Odessa, Ukraine <https://gamow.odessa.ua/wp-content/uploads/2021/08/gamow-abstracts-2021.pdf>
6. T. Yushkevych, et.al, “Application of the Laplace method for one-loop diagrams of elastic scattering of protons in the framework of multiparticle fields”, International Conference of Students and Young Researchers in Theoretical and Experimental Physics “HEUREKA-2021”, May 18-20, 2021. Lviv, Ukraine <https://physics.lnu.edu.ua/conferences/heureka2021/>
7. Yushkevich T., Potiienko O., Neboga H., “Modeling the interference effects in elastic hadrons scattering with the Monte Carlo event generator based on Laplace’s method”, 20-th Gamow International Astronomical Conference-School “Astronomy and beyond: astrophysics, cosmology and gravitation, high energy physics, astroparticle physics, radioastronomy and astrobiology” 9–16 August, 2020, Odessa, Ukraine <http://gamow.odessa.ua/wp-content/uploads/2020/10/>

8. D.A. Ptashynskyi, T.V. Yushkevich, I.V. Sharph, V.D. Rusov, et.al, “Multi-particle fields on a subset of simultaneity”, New Trends in High-Energy Physics, May 12-18, 2019, Odessa, Ukraine. <https://indico.bitp.kiev.ua/event/1/contributions/62/>

Окрім висвітлених вище в тезисах конференцій, 1 статтю додано до arXiv (Sharph I. V., Zelentsova T. M., Chudak N. O., Potiienko O. S., Ptashynskiy D. A., Merkotan K. K. Yushkevich T. V. and Milieva, A. O., Laplace method for the simplest diagrams of elastic scattering of scalar particles, <https://arxiv.org/abs/2003.07321>), а інша стаття наразі готується до публікації у журналі, що входить у базу Scopus. Також основні положення і результати дисертації доповідалися й обговорювалися на International School “Frontiers in Nuclear and Hadronic Physics 2025” (Galileo Galilei Institute, Florence, Italy, 2025), 49th SLAC Summer Institute (California, USA, 2021), Amplitudes 2021 (Niels Bohr International Academy and the Niels Bohr Institute, Penn State, USA, 2021), IrisHEP Fellowship program (Princeton University, USA, 2022) та Міжнародній конференції студентів і молодих науковців з теоретичної та експериментальної фізики “Еврика-2020” (Львів, 2020).

Abstract

Yushkevych T. V. Perturbative and Non-Perturbative Aspects of Hadronic Physics — Qualification scientific work presented as a manuscript.

Dissertation for the degree of Candidate of Physical and Mathematical Sciences (PhD) in the specialty 104 — Physics and Astronomy (10 — Natural Sciences). — Odesa National Polytechnic University, Ministry of Education and Science of Ukraine, Odessa, and University of Turin, Turin (Italy), 2025.

This thesis presents a combined theoretical and phenomenological investigation of QCD dynamics, focusing on two complementary aspects.

First, the method of multiparticle fields is applied to construct models of proton (antiproton) scattering processes. To approximate the contribution of loop diagrams within this framework, the Laplace method is used, providing a practical yet improvable computational tool. This work demonstrates that the method of multiparticle fields allows for the construction of a dynamical model to describe proton scattering processes as a system of quarks, achieving qualitative agreement with experimental observations. In particular, the application of the Laplace method offers an approximate, albeit computationally demanding, approach for evaluating loop diagrams in these models. It is shown that the experimentally observed non-monotonic behavior of the differential cross section for elastic proton scattering as a function of the squared momentum transfer arises from spin effects, which, when incorporated into both tree-level

and simple loop diagrams, lead to qualitative agreement with experimental data.

Second, an affinity-based methodology is developed to explore the kinematic transition between transverse momentum dependent (TMD) and collinear factorization regimes in semi-inclusive deep inelastic scattering (SIDIS). For this purpose, a detailed kinematic assessment of the SIDIS phase space is proposed using the "affinity" tool. This methodology enables the identification of regions dominated by TMD factorization, collinear factorization, as well as the matching region where both regimes overlap, providing new opportunities to investigate the factorization scheme and the hadronization mechanism. The transition from bin-averaged to event-by-event affinity calculations preserves essential correlations among non-perturbative parameters, leading to a more reliable classification of kinematic regions. The analysis shows that the TMD region is broader than previously expected, especially within the JLab12 kinematics, while the collinear region becomes well accessible only at higher energies, like those of JLab22 and the Electron-Ion Collider. Within the affinity framework, the identification of distinct TMD, Collinear and Matching Regions becomes possible, providing new prospects for studying the interplay between factorization regimes.

The findings of this thesis work offer both theoretical insights and practical tools for future studies of nucleon structure and hard scattering processes. Recommendations for further development include incorporating higher-order loop diagrams into the multiparticle field framework, refining computational methods, extending affinity-based analyses to other

processes, and guiding experimental efforts through targeted kinematic cuts.

Key words: High-Energy Physics: phenomenology and experiment; Quantum Chromodynamics (QCD): partons, quarks and gluons; Hard Scattering Processes; Multi-Quark Scattering; Proton Elastic Scattering; Spin Physics; Loop Diagram Approximation; Inclusive and Semi-inclusive Deep Inelastic Scattering (DIS and SIDIS); Nucleon Structure; Transverse Momentum Dependence (TMD); Collinear and TMD Factorization; Phenomenological and mathematical modeling; Monte Carlo methods; Neural Networks; Nuclear reactions

List of Publications of the Author:

The main results of the dissertation are presented in 4 publications, 3 of which are indexed in the Scopus scientometric database (2 articles published in journals ranked in the first quartile (Q1) according to the SCImago Journal and Country Rank classification). Additionally, the results have been presented in 5 conference abstracts at international scientific conferences.

- *journal publications:*

1. Chudak N., Sharph I., Potiienko O., Milieva A., Merkotan K., Yushkevych T. and Rusov V., Laplace's method for elastic scattering diagrams within multiparticle fields model, Phys. Review D 106, 3 (2022) DOI:<https://doi.org/10.1103/PhysRevD.106.036023>

2. O. Potiienko, K. Merkotan, N. Chudak, D. Ptashynskiy, T. Zelentsova, T. Yushkevich, I. Sharph, and V. Rusov, New method of accounting for interference contributions within a multiperipheral model, *Physical Review D* 101, 7 (2020) DOI:<https://doi.org/10.1103/PhysRevD.101.076021>
3. D.A. Ptashynskiy, T.M. Zelentsova, N.O. Chudak, K.K. Merkotan, O.S. Potiienko, V.V. Voitenko, O.D. Berezovskiy, V.V. Opyatyuk, O.V. Zhavorova, T.V. Yushkevich, I.V. Sharph, V.D. Rusov, Multiparticle fields on the subset of simultaneity, *Ukrainian Journal of Physics* 64(8), 732 (2019). DOI: <https://doi.org/10.15407/ujpe64.8.732>

Approbation of the results of the dissertation

- *conference abstracts:*
4. T. Yushkevych, M. Boggione, A. Prokudin, "Affinity tool. Comparison between JLab12 and JLab22", "Science at the Luminosity Frontier: Jefferson Lab at 22 GeV" (Frascati, Italy, 2024) (<https://agenda.infn.it/event/39742/contributions/247698/>)
 5. T. Yushkevych, I. Sharph, et.al, "Laplace method for single-loop diagrams of elastic proton scattering", 21-th Gamow International Astronomical Conference-School "Astronomy and beyond: astrophysics, cosmology and gravitation, high energy physics, astroparticle physics, radioastronomy and astrobiology", August 15-21, 2021 Odessa, Ukraine <https://gamow.odessa.ua/wp-content/uploads/2021/08/gamow-abstracts-2021.pdf>

6. T. Yushkevych, et.al, “Application of the Laplace method for one-loop diagrams of elastic scattering of protons in the framework of multiparticle fields”, International Conference of Students and Young Researchers in Theoretical and Experimental Physics “HEUREKA-2021”, May 18-20, 2021. Lviv, Ukraine <https://physics.lnu.edu.ua/conferences/heureka2021/>
7. Yushkevich T., Potiienko O., Neboga H., “Modeling the interference effects in elastic hadrons scattering with the Monte Carlo event generator based on Laplace’s method”, 20-th Gamow International Astronomical Conference-School “Astronomy and beyond: astrophysics, cosmology and gravitation, high energy physics, astroparticle physics, radioastronomy and astrobiology” 9–16 August, 2020, Odessa, Ukraine <http://gamow.odessa.ua/wp-content/uploads/2020/10/>
8. D.A. Ptashynskyi, T.V. Yushkevich, I.V. Sharph, V.D. Rusov, et.al, “Multi-particle fields on a subset of simultaneity”, New Trends in High-Energy Physics, May 12-18, 2019, Odessa, Ukraine. <https://indico.bitp.kiev.ua/event/1/contributions/62/>

In addition to the publications and conference abstracts listed above, one article has been uploaded to arXiv (Sharph I. V., Zelentsova T. M., Chudak N. O., Potiienko O. S., Ptashynskiy D. A., Merkotan K. K. Yushkevich T. V. and Milieva, A. O., Laplace method for the simplest diagrams of elastic scattering of scalar particles, eprint arXiv:2003.07321, <https://arxiv.org/abs/2003.07321>), and another article is currently

being prepared for submission to a Scopus-indexed journal. The main results and conclusions of the dissertation have also been presented and discussed at the International School “Frontiers in Nuclear and Hadronic Physics 2025” (Galileo Galilei Institute, Florence, Italy, 2025), the 49th SLAC Summer Institute (California, USA, 2021), the “Amplitudes 2021” workshop (Niels Bohr International Academy and Niels Bohr Institute, Penn State, USA, 2021), IrisHEP Fellowship program (Princeton University, USA, 2022) and the International Conference of Students and Young Scientists in Theoretical and Experimental Physics “Eureka-2020” (Lviv, Ukraine, 2020).

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Chapter 1

Introduction

This dissertation presents a combined theoretical and phenomenological investigation into the dynamics of Quantum Chromodynamics (QCD), with a particular focus on nucleon structure and factorization mechanisms in high-energy scattering processes.

The research addresses two complementary topics: the modeling of proton (and antiproton) scattering using the method of multiparticle fields, and the development of a novel affinity-based methodology to analyze the transition between transverse momentum dependent (TMD) and collinear factorization regimes in semi-inclusive deep inelastic scattering (SIDIS).

Motivation for the choice of the research topic. The motivation stems from two central challenges in modern hadronic physics. First, understanding elastic and inelastic hadronic scattering processes remains an open and active area of research, where effective models grounded in QCD principles are essential for interpreting experimental data. Previously, these processes were not described by a dynamic model. Our

objective is to provide a description of the observed non-monotonic behavior of the differential cross section for elastic scattering as a function of the squared transferred four-momentum, based entirely on a dynamic approach. By a dynamic model, we refer to a framework constructed from fundamental physical principles, beginning with the formulation of the Lagrangian, followed by deriving the corresponding dynamical equations, applying the quantization procedure, and solving the equations that govern the dynamics of the relevant relativistic quantum system within Fock space. Such a description of the effects will help to more accurately understand the results of experiments. Second, the theoretical consistency and practical applicability of factorization theorems—especially the interplay between transverse momentum dependent (TMD) and collinear factorization—are critical to the interpretation of SIDIS data and the extraction of parton distribution functions, in both collinear and TMD regimes.

Both problems are central to current international efforts in hadron physics, as pursued at facilities such as JLab, LHC, and the future Electron-Ion Collider (EIC). By addressing the modeling of scattering amplitudes and the precise delineation of factorization regimes, this dissertation contributes to foundational and phenomenological aspects of QCD.

Research Goals and Objectives. The main goal of this thesis is twofold: First, to develop and test a dynamical model of proton scattering based on the method of multiparticle fields, using the Laplace approximation to evaluate loop contributions. Second, to construct and apply an affinity-based classification framework to study the transition between

TMD and collinear factorization regimes in SIDIS, possibly achieving some degree of insight in the intermediate region, where a continuous and smooth matching between the predictions obtained from the collinear and TMD factorization schemes is expected.

More specifically, the main objectives are the following:

- To investigate spin effects in elastic proton scattering and their impact on the differential cross section.
- To explore the advantages offered by the Laplace method for approximating loop diagrams in hadronic processes, while also assessing the limitations.
- To define kinematic boundaries between factorization regimes using the affinity tool, moving from a methodology based on a bin-by bin analysis to a newly developed strategy based on an event-by-event analysis.
- To assess the applicability of the proposed affinity methodology across different experimental energy scales.

Research Methodologies. The research combines analytical modeling and numerical computation. The method of multiparticle fields is employed to construct effective models of scattering amplitudes at both tree level and loop corrections. The Laplace approximation is applied to evaluate loop diagrams, providing an analytically tractable and computationally feasible method for estimating quantum corrections.

For the SIDIS analysis, a kinematic classification technique based on the affinity between experimental data and theoretical assumptions is developed. This methodology is implemented through numerical simulations and Monte Carlo techniques to analyze multidimensional phase space distributions and classify data points according to their compatibility with TMD or collinear factorization.

The choice of methodologies is justified by their capacity to capture essential features of QCD dynamics, in both perturbative and non-perturbative regimes, to provide physically interpretable outcomes, and to ensure the reliability of the resulting models and classifications.

Scientific Novelty. The dissertation includes several original contributions.

- The application of multiparticle field theory - the first dynamical approach - to model proton-proton and proton-antiproton scattering, including the use of the Laplace method to approximate loop diagrams, represents a novel computational approach in this context.
- The demonstration of spin effects as the source of non-monotonic behavior in the differential elastic cross section provides a new physical interpretation within an effective model framework.
- The development of the affinity-based methodology to map TMD, collinear, and matching regions in SIDIS phase space is a conceptual and technical innovation. The transition to event-by-event affinity classification improves the resolution of kinematic boundaries and preserves correlations in the data that are otherwise averaged out.

- The findings reveal that the TMD regime extends beyond previously assumed bounds, especially at moderate energies (e.g., JLab12), suggesting a revision of common kinematic assumptions used in global fits and experimental analyses.

Personal Contribution of the Applicant. The results presented in this dissertation are the original work of the author, while more general methodologies have been developed in co-authorship (with the supervisors and their collaborators). In cases where collaborative publications or joint projects have contributed to the development of specific ideas (e.g., related to the affinity tool), the applicant’s personal contribution is clearly identifiable in the conceptual formulation, implementation, and analysis of results. When appropriate, references to publications co-authored by the supervisors are provided.

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Practical Significance. The methodologies developed in this thesis, offer practical tools for data selection and analysis in future experimental programs. These tools can assist in optimizing kinematic cuts, validating

factorization assumptions, and guiding the extraction of non-perturbative inputs in global fits. The Laplace-based approach to loop contributions also has potential for broader applications in effective QCD modeling.

Approbation of the results of the dissertation. The main principles and results of the dissertation were reported and discussed at the workshop “Science at the Luminosity Frontier: Jefferson Lab at 22 GeV” (Frascati, Italy, 2024), the International School “Frontiers in Nuclear and Hadronic Physics 2025” (Galileo Galilei Institute, Florence, Italy, 2025), the 49th SLAC Summer Institute (California, USA, 2021), the “Amplitudes 2021” workshop (Niels Bohr International Academy and Niels Bohr Institute, Penn State, USA, 2021), and the International Conference of Students and Young Scientists in Theoretical and Experimental Physics “Eureka-2020” (Lviv, Ukraine, 2020).

Outline of the thesis

In this work, I focus on two distinct but interconnected problems: Semi-Inclusive Deep Inelastic Scattering (SIDIS) and elastic hadron scattering, employing specialized theoretical tools suited to each process. **Chapter 2** describes the main models

For SIDIS, the Parton Model provides a robust foundation, offering a well-established framework to interpret high-energy quark-gluon interactions within factorizable cross sections. However, in the case of elastic hadron scattering, I adopt the Multi-Particle Field Model, a dynamical approach developed over years within our research group. This model demonstrates superior descriptive power compared to the Parton Model

for elastic processes, as it inherently incorporates non-perturbative effects and interparticle correlations critical at lower momentum transfers. A detailed justification for this choice, along with the model’s theoretical underpinnings and predictive capabilities, is presented in **Chapter 3**.

A recurring challenge in hadronic physics is the separation of calculable perturbative contributions from non-perturbative dynamics in experimental observables. To address this, I have developed a Neural Network-based tool designed to systematically classify datasets in SIDIS according to their compatibility with theoretical frameworks. This tool aids in isolating regions of the cross section amenable to perturbative treatment from those requiring non-perturbative input, thereby enhancing the precision of QCD analyses. The methodology, validation, and applications of this approach are discussed in **Chapter 4**.

Chapter 2

Hadron structure: from the parton model and Regge theory, to QCD factorization theorems

The study of hadronic processes lies at the heart of quantum chromodynamics (QCD), where the dynamics of quarks and gluons give rise to the rich phenomenology of strongly interacting matter. Theoretical approaches to these processes are broadly divided into perturbative and non-perturbative methods, each applicable in distinct regimes and often requiring careful interplay to achieve a complete picture. At high momentum transfer or energy scales much larger than the QCD confinement scale, the smallness of the running coupling constant allows for perturbative expansions in powers of α_s . In this regime, observables such as inclusive cross sections in deep-inelastic scattering or jet production in hadron collisions can be systematically computed using Feynman diagrammatic techniques, often supplemented by factorization theorems that separate short-distance perturbative dynamics from long-distance

non-perturbative matrix elements like parton distribution functions or fragmentation functions.

However, QCD's non-perturbative nature becomes unavoidable when probing processes sensitive to the confinement scale, where α_s grows large and quarks and gluons no longer have the appropriate degrees of freedom. In these cases, non-perturbative methods must be employed.

The boundary between perturbative and non-perturbative QCD is not always sharp, and many physical observables require a synthesis of both approaches. For instance, the description of hadron structure in high-energy scattering involves perturbative evolution equations for parton densities matched onto non-perturbative input from lattice QCD or global fits to data. Understanding the interplay between these methods is essential for advancing our theoretical control over QCD across all energy scales, from the structure of nucleons to the properties of quark-gluon plasma.

The study of hadronic processes encompasses a diverse range of theoretical approaches, each tailored to specific kinematic regimes and physical observables. While no single framework universally describes all phenomena, the complementary strengths of different models allow for a deeper understanding of QCD dynamics across various scales.

2.1 Parton Model

The Parton Model was firstly introduced by Richard Feynman in 1969 [1] to describe the internal structure of hadrons in terms of point-like constituents called *partons*. These *partons* can be identified with quarks and gluons in QCD and they carry fractions of the hadron's momentum, especially in high-energy collisions. It was proposed during the era of experiments at SLAC where deep inelastic scattering data [2] revealed a particular scaling behaviour of the structure functions, known as Bjorken scaling [3]. This parton picture was able to elegantly explain these results using relativistic constituents inside a hadron.

The Parton Model is based four main assumptions. Firstly, the hadron is considered in infinite momentum frame (IMF) where it has very large longitudinal momentum which simplifies the kinematics and freezing parton interactions during the scattering. Secondly, partons behave as free particles during the short time of interaction with the probing lepton due to the asymptotic freedom in QCD. During the interaction, the lepton interacts with a single parton at a time and interference from other partons is negligible. And the last one - each parton carries a fraction x (Bjorken variable) of the hadron's longitudinal momentum.

2.2 Parton Model application in SIDIS

The use of Parton Model has a wide range, but we are going to make an example on SIDIS process fundamentally based on the Parton Model

since that is a part of the research this thesis focuses on.

In electron-nucleon scattering processes, an incident electron carrying four-momentum l interacts with a nucleon (of mass M , four-momentum P , and spin S) through virtual photon (or weak boson) exchange with four-momentum transfer $q = l - l'$, where l' represents the final-state electron momentum, P_h is identified hadron in the final state and P_X is not detected remnants of the nucleon. Simple scheme of this process is shown in Fig. 2.1.

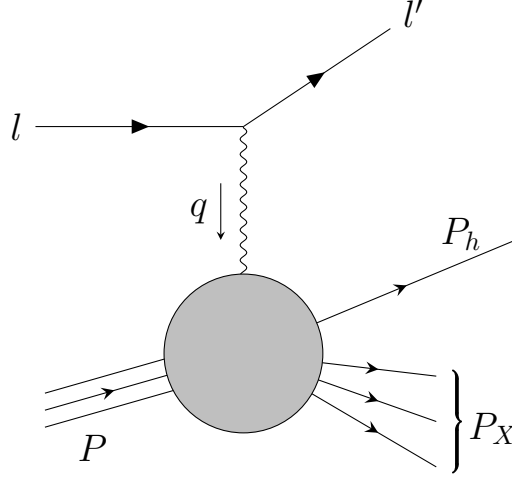


FIGURE 2.1: Semi-inclusive deep inelastic scattering process

Introducing variables

$$x_{Bj} = \frac{Q^2}{2P \cdot q}, \quad z_h = \frac{P \cdot P_h}{P \cdot q}, \quad y = \frac{P \cdot q}{P \cdot l}, \quad (2.1)$$

we can write an equation for SIDIS cross-section:

$$\frac{d\sigma}{dx_{Bj} dy dz_h d^2p_T} \propto \sum_q e_q^2 \underbrace{f_q(x, p_T)}_{\text{TMD PDF}} \underbrace{D_q^h(z, p_T)}_{\text{TMD FF}}, \quad (2.2)$$

where $Q^2 = -q^2$ is the momentum transfer squared, x_{Bj} - the Bjorken scaling variable, z_h - the hadron momentum fraction, y - inelasticity and $f_q(x, p_T)$ and $D_q^h(z, p_T)$ are Transverse Momentum-Dependent (TMD) Parton Distribution Function (PDF) and Fragmentation Function (FF) respectively. TMD PDFs are describing a probability to find quark q with momentum (x_{Bj}, p_T) and TMD FFs are describing a probability for quark q to produce hadron h with (z, p_T) . Factorizations theorems allow us to separate our cross section into two parts: hard part when Q^2 is large and photon-quark interaction is perturbative, and soft part which encodes non-perturbative QCD effects through TMDs and FFs and they are determined from experiment.

2.3 QCD Factorization

A modern and powerful framework for investigating hadron structure within Quantum Chromodynamics (QCD) is provided by factorization theorems. The primary objective of factorization is to systematically separate the contributions arising from different physical scales—most notably, to disentangle the short-distance (ultraviolet) dynamics, which are amenable to perturbative calculations, from the long-distance (infrared) dynamics, which are inherently nonperturbative.

The detailed formalism of factorization will not be discussed here; instead, I will briefly outline its fundamental features, following Refs. [4, 5].

Factorization methods allow us to decompose the contributions to a scattering process into two distinct categories: the short-distance contributions, which are perturbatively calculable, and the long-distance contributions, which are nonperturbative and depend on small energy scales.

These contributions are identified by analyzing the kinematical regions that dominate the Feynman integral for the process, denoted as $G\left(\frac{p_i^\nu}{\mu}, \frac{m}{\mu}\right)$, where p_i^ν are the external momenta, constrained by

$$p_i \cdot p_j = Q^2 \eta_{ij}, \quad p_i^2 = \zeta_i m^2, \quad (2.3)$$

with $Q \gg m$. This integral can be generically expressed as the integration over L loop momenta acting on a product of I Feynman propagators, multiplied by a function N which encodes numerator factors associated with internal lines.

The goal is to identify the regions in the loop momentum space l^μ that contribute at leading power in Q^2 , called leading regions. By keeping the hard scale $Q \gg m$, the short-distance contributions arise from regions where the internal momenta $\{k_j^\mu\}$ are of order Q , so that the corresponding lines are far off-shell by at least $O(Q^2)$. At leading power in Q , particle masses can be neglected, as the corresponding errors are suppressed by $O(m/Q)$, and are asymptotically smaller than perturbative corrections of order $\alpha_s(Q)$.

Once the factorization scale μ is chosen to be of order Q , one can distinguish the two types of contributions: short-distance contributions involve

fixed, non-vanishing propagator denominators, while long-distance contributions are characterized by vanishing denominators. The latter lead to divergences associated with pinched singular surfaces (PSS), where internal lines are trapped between poles, making the singularity unavoidable by contour deformation.

At this stage, factorization identifies three relevant momentum regions:

- The hard region, where all loop momenta have large components and the corresponding propagators are far off-shell;
- The collinear regions, where momenta become aligned with external hadrons and virtualities are small;
- The soft region, where momenta are small in all components.

The contributions from the collinear and soft regions are universal and process-independent: they are encoded in nonperturbative functions such as parton distribution functions (PDFs), fragmentation functions (FFs), or transverse momentum dependent distributions (TMDs), depending on the kinematic regime and observable of interest. These functions absorb the long-distance physics, while the hard part encodes the short-distance scattering and can be computed order-by-order in perturbation theory.

In summary, factorization theorems provide a rigorous and practical framework to compute QCD observables by splitting the problem into a hard, perturbative sector and a soft, nonperturbative sector, each governed by different degrees of freedom but consistently matched at a

common scale. This separation is the foundation of modern QCD phenomenology and underpins a vast range of applications from deep inelastic scattering to hadron collider processes.

2.4 Reggeized Theory

Reggeized theory (or **Regge theory**) is a theoretical framework that was developed by T. Regge in 1959 and describes high-energy scattering processes in particle physics particularly where perturbative methods fail. It emerged from the observation that certain particles (e.g., Pomeron [6] in QCD) can be treated as composite objects. Regge extended the concept of partial wave analysis into the complex angular momentum plane which lead to a new interpretation of resonances and scattering processes. The core idea of Reggeized theory is to replace the sum over partial waves with an analytic computation into the complex angular momentum plane [7]. "Reggeization" replaces Feynman propagators with Regge-pole-dominated amplitudes and doing so, it captures the leading logarithmic behavior of high-energy scattering. This theory uses Regge poles and trajectories as cornerstones of high-energy physics, providing a framework to classify hadrons and describe scattering amplitudes in non-perturbative regime.

2.4.1 Regge poles and trajectories

A scattering amplitude $f(\theta)$ for a central potential in partial wave analysis is expanded in terms of Legendre polynomials:

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) a_{\ell} P_{\ell}(\cos \theta), \quad (2.4)$$

where ℓ is the angular momentum quantum number and a_{ℓ} are the partial wave amplitudes. If we treat ℓ as a complex variable and analytically continuing a_{ℓ} to a function $a(\ell)$, these continuations will reveal singularities in the complex ℓ -plane, which are known as **Regge poles**. These poles are associated with bound states, exchange forces and resonances.

Consider the Schrödinger equation for a potential in the centre-of-mass system (where $\hbar = c = 2\mu$ and μ is the reduced mass of the two-body system)

$$\left(\nabla^2 + k^2 - \frac{\ell(\ell + 1)}{r^2} - V(r) \right) \psi(r) = 0, \quad (2.5)$$

where k is the wave number. The centre-of-mass scattering angle θ and k are related to the energy E and to the Mandelstam variables by the relations: $k^2 = E = s$, $-t = 2E(1 - \cos \theta)$.

For (2.5) we can identify two classes of solutions: solution $\varphi(\ell, k, r)$ that in the origin behaves like $\varphi(\ell, k, r) \approx r^{\ell}$ and solution $f(\ell, k, r)$ that at infinity behaves like $f(\ell, k, r) \approx e^{-ikr}$ [8]. The potential in this equation is given by a Yukawian representation

$$V(r) = \int_{m>0}^{\infty} d\mu \sigma(\mu) \frac{\exp[-\mu r]}{r}. \quad (2.6)$$

Introducing a so-called Jost function as the Wronskian of the two previous solutions $f(\ell, k) \equiv W[f(\ell, k, r), \varphi(\ell, k, r)]$ and making a phase-space shift $\delta(\ell, k)$, the partial-wave scattering amplitude becomes

$$A_\ell(k) = \frac{S(\ell, k) - 1}{2ik}, \quad (2.7)$$

where $S(\ell, k)$ is the partial-wave S -matrix and in terms of the Jost functions $S(\ell, k) = \frac{f(\ell, k)}{f(\ell, -k)} e^{i\pi\ell}$ [9].

A very interesting property of the scattering amplitude was noticed by applying the crossing symmetry, which shows that the physical roles of s and t are reversed, which follows from the connection between the poles and the asymptotic behavior of the scattering amplitude [10]. Also it was found by Regge, that due to the vanishing of $f(\ell, -k)$ may lead to the singular behavior in the scattering amplitude. The position of these singularities (poles) can be described by functions $\alpha(s)$ or $\alpha(t)$ that are called **Regge trajectories**.

Regge trajectory $\alpha(t)$ is a function that relates the spin J of a resonance to its squared mass m^2 :

$$J = \alpha(m^2) = \alpha_0 + \alpha' m^2 \quad (2.8)$$

where α_0 and α' are constants that are empirically determined and α' is related to the slope of the trajectory.

These trajectories are approximately linear and group resonances (e.g., mesons or baryons) into families that share internal quantum numbers

such as isospin or parity. Regge trajectories are observed in meson spectra (ρ , ω , K^* families), baryon spectra (Δ , N^* trajectories) and high-energy scattering (pp , πp cross-sections) [11–17]. Their linear behavior is supported by experimental data and one of the examples is shown on Fig. 2.2 [18].

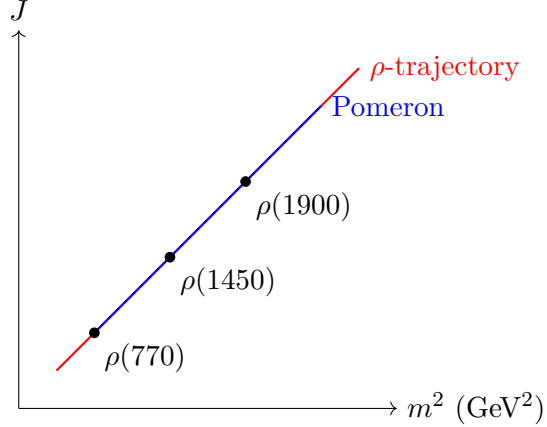


FIGURE 2.2: Chew-Frautschi plot for the ρ -meson trajectory.

In scattering theory, Regge poles describe how the scattering amplitude behaves at high energies and fixed momentum transfer. This condition is called a **Regge limit** ($s \rightarrow \infty$, t fixed).

The process of **reggeization** refers to the replacement of a single-particle exchanges (e.g., vector mesons propagators) with the entire Regge trajectories. For example, a scalar particle propagator

$$\frac{1}{t - m^2} \quad (2.9)$$

can be Reggeized by replacing it with

$$\mathcal{P}_{Regge}(s, t) = \eta(t)\beta(t) \left(\frac{s}{s_0} \right)^{\alpha(t)} \quad (2.10)$$

where $\eta(t)$ is a signature factor, $\beta(t)$ is a residue and s_0 is a scale parameter.

Because of this replacement, the scattering amplitude in the Regge limit behaves as

$$A(s, t) \sim \beta(t) s^{\alpha(t)} \quad (2.11)$$

This power-law behavior differs from the exponential to logarithmic dependence in other regimes and is a key prediction of Reggeized theory.

2.4.2 Elastic scattering processes in Reggeized theory

The starting point of all constructions within the Reggeon theory is the amplitude of elastic scattering, schematically represented by the diagram in Fig. 2.3.

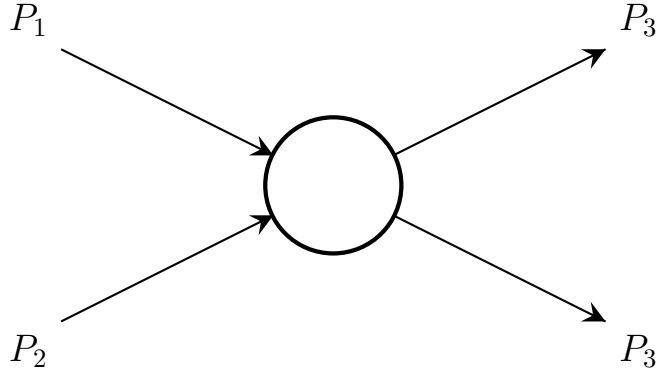


FIGURE 2.3: A schematic diagram of the elastic process

The scattering amplitude corresponding to the elastic process in Fig. 2.3 is a function of the four vectors of energy – momentum of all four particles shown in Fig. 2.3. We will denote this function by $T(P_1, P_2, P_3, P_4)$.

At the same time, four momenta, on which the scattering amplitude depends, satisfy the conditions on the mass shell

$$P_k^2 = m_k^2, \quad k = 1, 2, 3, 4 \quad (2.12)$$

Here P_k^2 are the scalar squares of the corresponding four vectors in the Minkowski space: $P_k^2 = (P_k^0)^2 - (P_k^1)^2 - (P_k^2)^2 - (P_k^3)^2$, and the masses of the particles are denoted by m_k .

Within the framework of perturbation theory, a property called crossing symmetry can be proved for the scattering amplitude corresponding to any process, including the one shown in Fig. 3.1. This property consists in the fact that the amplitude of the process in Fig. 3.1 and the process obtained from it by exchanging the places of particles with four-momenta P_1 and P_2 with simultaneous replacement of these particles by antiparticles, are identical functions of s and t . At the same time, the functions $T'(s, t)$ and $T(s, t)$ will be the scattering amplitudes for "their" processes under the conditions that pairs of values s and t are also substituted in them from different regions.

Taking into account mentioned above, it is needed to examine the final form of the scattering amplitude expression for the process of elastic proton collision and analyze it.

The formula for the limit for large values of s takes the form:

$$\begin{aligned}
T(s, t) \xrightarrow{s \rightarrow \infty} & \sum_{\sigma=-1}^1 -\frac{1}{4i} \int_F \frac{T'^{\sigma}(l, t) \exp(-i\pi l) + \sigma}{2 \sin(\pi l)} \left(1 + \frac{2s}{t - 4m^2}\right)^l dl - \\
& - \frac{\pi}{2} \sum_{\sigma=-1}^1 \sum_{j=1}^N r_j^{\sigma}(t) \frac{\exp(-i\pi \alpha_j^{\sigma}(t)) + \sigma}{2 \sin(\pi \alpha_j^{\sigma}(t))} \left(1 + \frac{2s}{t - 4m^2}\right)^{\alpha_j^{\sigma}(t)}
\end{aligned} \tag{2.13}$$

For those values of s , in the expression $1 + \frac{2s}{t-4m^2}$ the value of 1 can be neglected, and the factor $\frac{2s}{t-4m^2}$ is usually replaced by a constant $\frac{1}{s_0}$ due to the fact that the scattering amplitude is significantly different from zero in the small region of the t parameter change. At the same time, s_0 is usually considered to be of the order of 1 GeV^2 . In addition, the background integral will be small for large values of s . The contribution from this integral is usually neglected. Then entering the notation

$$\eta(\alpha_j^{\sigma}(t)) = -\frac{\exp(-i\pi \alpha_j^{\sigma}(t)) + \sigma}{\sin(\pi \alpha_j^{\sigma}(t))} \tag{2.14}$$

the expression (2.13) becomes:

$$T(s, t) \xrightarrow{s \rightarrow \infty} \frac{\pi}{4} \sum_{\sigma=-1}^1 \sum_{j=1}^N r_j^{\sigma}(t) \eta(\alpha_j^{\sigma}(t)) \left(\frac{s}{s_0}\right)^{\alpha_j^{\sigma}(t)} \tag{2.15}$$

The value $\eta(\alpha_j^{\sigma}(t))$ introduced by relation (2.14) is usually called the signature multiplier. Expression (2.15) is called the Reggeon asymptotics of the scattering amplitude.

This expression is important first of all from the point of view that it is a reference point for all further considerations within the framework of

Reggeon theory. In particular, this is manifested in the fact that many approximations made during the consideration of elastic processes are mostly "justified" by the fact that they lead to a relationship of the type (2.15).

Chapter 3

Laplace method for elastic hadron scattering

Elastic scattering of protons has been the subject of experimental and theoretical research for several decades. The experimental data (Fig. 3.1) can be reproduced within the framework of traditional schemes supplemented by the assumption of an intercept $\alpha(0) > 1$ and taking into account unitary corrections (absorption, rescattering) to a simple pole. These corrections are not small, and therefore the predictions of various approaches such as quasi-potential approach, u-matrix formalism, eikonal approach, reggeon field theory and others, used for their calculation differ greatly.

A large amount of experimental results has been accumulated [20], but a dynamic theory based on fundamental physical principles still does not exist [21]. Usually, the description of such processes is built on the basis of phenomenological approaches [22], models that are different variants of the Reggeon theory [9, 23], or geometric models [24–26]. Such approaches

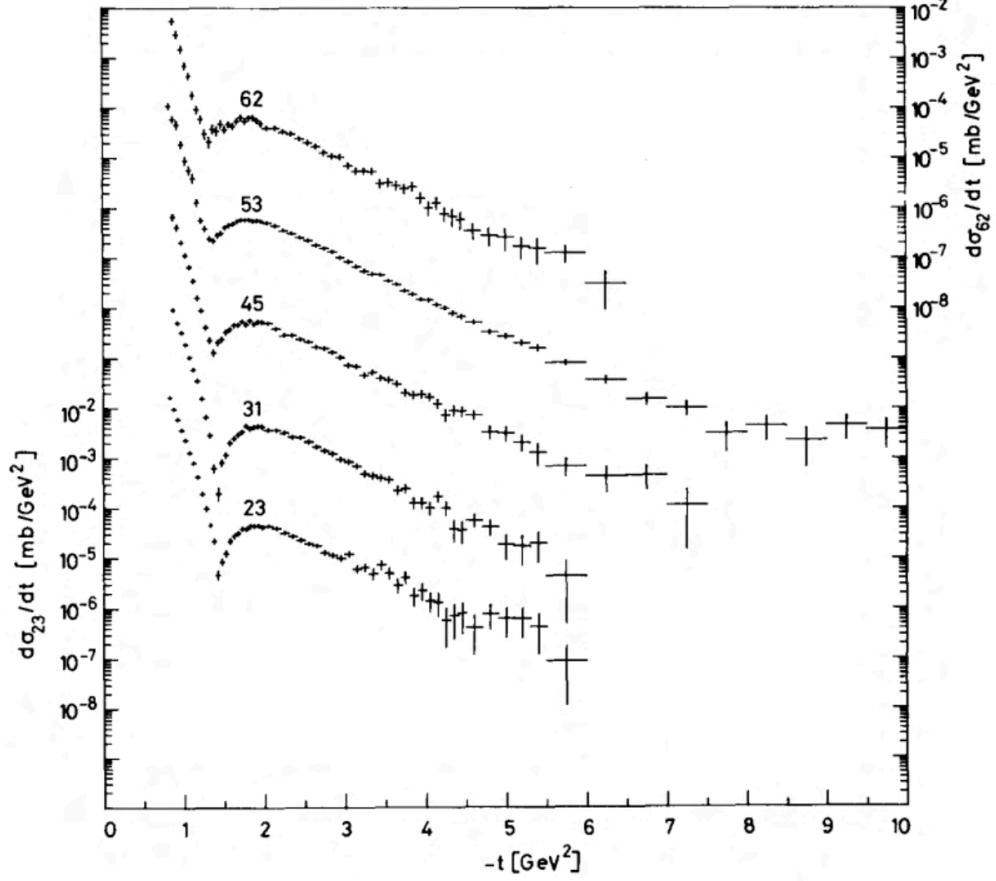


FIGURE 3.1: The dependence of the differential cross section $\frac{d\sigma_{el}}{dt}(t)$ on the square of the transmitted four-momenta t , taken from the work [19]

are based on certain assumptions, which are not only not consequences of fundamental physical principles, but may even contradict them. In particular, in Reggeized models [27–29], on which regional models are based, there is an assumption that the main contribution to the integrals that determine the observed values is made by the multi-Regge region [30–33]. The use of this assumption leads to the fact that when calculating the scattering cross sections [23], the integration is performed over the phase space region, the different points of which are characterized by significantly different values of the particle energy-momentum in the final state [34]. Among the experimental data on proton-proton (antiproton) scattering, the most attention in the theoretical description is given to

the description of elastic scattering [35]. This is usually justified by the fact that this description should be simpler than the description of inelastic processes due to the smallest possible number of particles in the final state and because the scattering amplitudes are functions of only two Lorentz-invariant quantities [?]. The description of inelastic processes is tried to be obtained from the description of elastic scattering using the rules of cutting multi-region Abramovsky-Hribov-Kancheli diagrams [36–39]. However, with this approach, assumptions about the type of multi-region peak [40, 41] are added to the assumptions of the Reggeized models, as well as the introduction of such an extra-experimental quantity as the scattering cross section with the exchange of a certain number of Reggeons. But all these assumptions are aimed at justifying the dependence of the amplitudes of elastic scattering on the Mandelstam variable s , while the dependence on another variable t remains undefined. In particular, in the simplest one-Reggeon model, the dependence on t is included both in the trajectory of the Regge pole and in the excess. The trajectory is determined by the masses of t -channel resonances, but the remainder is a completely uncertain value. From the point of view of region diagrams, this excess is interpreted as the product of the interaction vertices of the region with hadrons [42–44], which are elastically scattered. The dependence of these peaks on t is given for the reasons of describing the experimentally measured differential sections of elastic scattering $d\sigma/dt$ with respect to t at small values of this parameter. Usually, the dependence of hadron-region vertices on t is postulated in the form of $\exp(-R^2|t|)$, where R is a fitting parameter called the Hadron

Reggeon radius [42].

This postulation makes it possible to reproduce the observed dependence of the differential cross-section of elastic scattering on t at small values of t , which is close to a straight line on a logarithmic scale along the axis of the cross-sections [45]. But firstly, this dependence is not direct over the entire interval of measurement in t , and secondly, it generally has a non-monotonic character with minimum and maximum points [19, 46–48]. It was not possible to describe these features within the framework of models with multi-Regge exchange of simple poles and the quasieikonal model of multi-Regge vertices [45, 49, 50]. For the exchange of simple Regge poles, it was possible to describe the non-monotonic behavior of the differential proton scattering cross section by the quadrature of the transmitted four-momentum t within the limits of the additive quark model [43] due to interference contributions from processes with the exchange of different numbers of Reggeons. Also within the framework of phenomenological approaches, this effect was reproduced in models with multiple Regge poles [51–53], but in such models the uncertainty of the dependence on t increases, because increasing the order of the pole increases the number of expansion coefficients in the Laurent series, the type of which has to be postulated. Also, the non-monotonic dependence of the differential section on the square of the transmitted four-momentum is reproduced within models with simple Regge poles of different signatures and the two-Reggeon sections generated by them [54].

In this chapter, the task of describing the considered effects of the non-monotonicity of the differential section of the elastic scattering $\frac{d\sigma}{dt}(t)$

from the square of the transmitted four-momentum based on a purely dynamic model, without any prior assumptions was set. By a dynamical model means a model that is built according to fundamental physical principles, starting from the setting of the Lagrangian and obtaining from it the dynamical equations, the quantization procedure and the solution of the equations that determine the dynamics of the corresponding relativistic quantum system in the Fock space. As such a model, we use the multiparticle field model [55].

3.1 Multiparticle field model

The model of multiparticle fields was proposed and applied in works [55–60]. In this model, it is shown that hadrons can be considered as quanta of a field defined on subsets of simultaneous events extracted from the tensor products of two (for mesons) or three (for baryons) Minkowski spaces. The domains of the values of these fields are the invariant subspaces of tensor products of bispinors, on which the scalar (for mesons) or bispinor (for protons) representation of the Lorentz group, as well as the trivial representation of the $SU_c(3)$ group corresponding to the colorlessness of hadrons, are realized. For such fields, a Lagrangian is constructed for the usual reasons and the calibration principle is applied to it. At the same time, local $SU_c(3)$ -invariance of the Lagrangian is achieved in a generalized way using calibration fields that are transformed by tensor representations of the Lorentz group and groups of internal symmetries. At the same time, the usual method of achieving calibration invariance

due to the lengthening of derivatives turns out to be a partial case of such a multi-particle method of achieving local invariance. In works [56, 58] it is shown that such tensor fields can be considered as describing the processes of creation and annihilation of bound states of gluons in a state of confinement. As you know, such bound states are called glueballs. At the same time, the same fields ensure the confinement of quarks in protons and mesons, as well as the interaction of quarks of different protons through the exchange of glueballs. As a result, we have a dynamic model for three-particle bispinor fields that interact through two-particle glueball fields. For such a model, the calculation of the observed quantities can be carried out in the usual way using R. Feynman's diagrammatic technique, because non-perturbative effects such as quark and gluon confinement have already been taken into account in the internal dynamics of quanta of multiparticle fields. At the same time, it naturally follows from the dynamic equations for the glueball field that they have a non-zero mass. This circumstance makes it possible to formulate a possible way of solving the problem of obtaining a finite value of the amplitude of elastic scattering due to strong interaction at $t = 0$. As is known, the full cross-section of proton scattering, or of a proton on an antiproton after exclusion of the electromagnetic interaction [26, 61] takes a finite value. As a result of the optical theorem [?], the scattering amplitude must take a finite value at $t = 0$. Thus, the problem of formulating such a model that would describe this circumstance arises. In all the phenomenological models described above, this finite value is simply postulated. In the QCD perturbation theory, we obtain an infrared singularity for scattering with

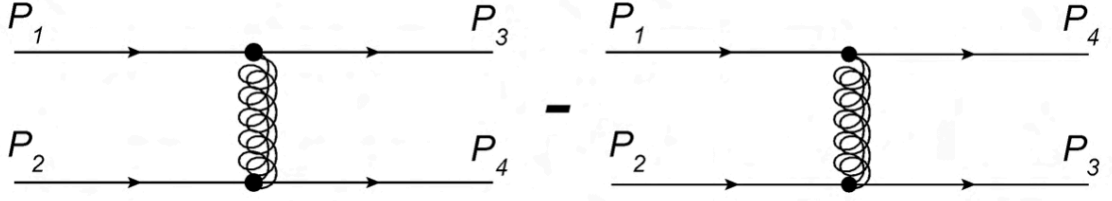


FIGURE 3.2: Polar loopless diagrams of proton elastic scattering. P_1, P_2 - four-momenta of incoming protons, P_3, P_4 - four-momenta of outgoing protons. Double lines correspond to the bound state of gluons - glueballs. The "minus" sign between the diagrams is due to the fact that protons obey Fermi-Dirac statistics

massless gluon exchange. It is clear that at $t = 0$ it is problematic to use the QCD perturbation theory, and the finite value of the elastic scattering amplitude may be a consequence of non-perturbative effects. But this reasoning does not answer the question of which non-perturbative effects and in what way lead to a finite value of the amplitude. The presence of a non-zero mass of the glueball, the existence of which is a consequence of non-perturbative effects, in the model of multiparticle fields leads to the finite value of the amplitude we need.

When describing inelastic scattering, it is possible to achieve agreement with the experiment at the level of a qualitative coincidence by limiting oneself only to consideration of loopless diagrams [62]. But when describing elastic scattering, consideration of loopless diagrams of the model of multiparticle fields shown in Fig. 3 does not provide a description of the differential cross section of scattering by the square of the transmitted four-momentum, even at the level of qualitative coincidence. Although already in such a simple model, the non-monotonic behavior of the section appears, this non-monotonicity is reduced only to the existence of a minimum point and does not reproduce the existence of a maximum point.

Therefore, the description of the experiment requires the calculation of more complex loop diagrams. The calculation of such diagrams is reduced to the calculation of multidimensional integrals over virtual four-momenta. Similar integrals were considered in the book [63] using R. Feynman's identity. However, in [63] only the location of integral features is considered, and the method of their calculation is not considered. Application of the Feynman identity simplifies the integrand expression due to the fact that instead of the product of several Feynman denominators, we find the power of one denominator. However, this significantly complicates the field of integration. Therefore, in this work, we consider an approximate calculation method based on a direct expression for the contribution of the loop diagram to the amplitude of elastic scattering. Similar integrals were also considered in [64] using the eikonal approximation. However, this approximation significantly distorts the pole structure of the integral expression. In addition, although its use allows to significantly reduce the dimension of the integral that must be calculated to find the eikonal in comparison with the original dimension of the integration, it does not solve the main problem associated with the calculation of such integrals. This problem does not lie in the large dimension of the integrals, but in the fact that, in fact, it is necessary to calculate the limit of the multidimensional integral when the parameters that ensure bypassing the poles of the integral expression approach zero. At the same time, the limit transition cannot be performed before the calculation of the integral, because in this case the poles fall inside the

region of integration and the integral loses its meaning. The need to calculate the integral up to the boundary transition limits the possibilities of applying numerical methods for calculating the integral. This happens because the limit transition will require calculation at small values of the parameters, at which the poles become close to the region of integration, which complicates the numerical calculation.

3.2 Laplace method

The description of the experiment described in the previous section with the use of the multiparticle field model requires the calculation of more complex loop diagrams. The calculation of such diagrams is reduced to the calculation of multidimensional integrals over virtual four-momenta. Similar integrals were considered in the book [63] using R. Feynman's identity. However, in [63] only the location of integral features is considered, and the method of their calculation is not considered. Applying the Feynman identity simplifies the integrand expression due to the fact that instead of the product of several Feynman denominators, we find the power of one denominator. However, this significantly complicates the field of integration. Therefore, in this work, we consider an approximate calculation method based on a direct expression for the contribution of the loop diagram to the amplitude of elastic scattering. Similar integrals were also considered in [64] using the eikonal approximation. However, this approximation significantly distorts the pole structure of the integral expression. In addition, although its use allows to significantly reduce the

dimension of the integral that must be calculated to find the eikonal in comparison with the original dimension of the integration, it does not solve the main problem associated with the calculation of such integrals. This problem does not lie in the large dimension of the integrals, but in the fact that, in fact, it is necessary to calculate the limit of the multidimensional integral when the parameters that ensure bypassing the poles of the integral expression approach zero. At the same time, the limit transition cannot be performed before the calculation of the integral, because in this case the poles fall inside the region of integration and the integral loses its meaning. The need to calculate the integral up to the limit transition limits the possibilities of applying numerical methods for calculating the integral. This happens because the limit transition will require calculation at small values of the parameters, at which the poles become close to the region of integration, which complicates the numerical calculation.

To overcome this problem, in this work we use the Laplace method [65]. This method was applied to the calculation of inelastic scattering processes in works [66–74] and made it possible to sufficiently efficiently calculate integrals over regions whose dimensions reach 100. However, the diagrams considered in these works corresponded to expressions in which the poles did not fall into the region at the limit transition integration and therefore the limit transition could be carried out before the calculation of the integral. The idea of using the Laplace method in this work is as follows. The considered problem arises due to the Feynman denominators, which are included in the expressions that are mapped to

the lines of the diagram. Renumber these lines in random order. We will present the expression in each denominator in the form $(z_a - i\varepsilon)^{-1}$. Here a is the number of the line to which the denominator corresponds, z_a corresponds to the line in terms of four-momentum of real and virtual particles, ε is a parameter that must be approximated to zero after calculating the integral. Then, choosing one or another subset of lines, we can equate their corresponding expressions z_a to zero and consider the system of equations obtained in this case. Then, if the system of some k equations turns out to be compatible, then it defines a subset of the region of integration on which the integrand behaves as ε^{-k} . Thus, the largest contribution to the integral will be made by the region in which the largest number of denominators will be zero. If this number is denoted as l , and the total number of integration variables is denoted by n , then from the corresponding system of equations, l variables can be expressed in terms of the remaining $n-l$. After that, it is convenient to introduce new integration variables instead of the first l integration variables, which are deviations from those values that satisfy the system of equations for l denominators. Then, at any values of $n-l$ variables, the module of the integral expression will have a sharp, due to its smallness, maximum at zero values of the first l variables. Then the Laplace method can be applied to integrate over these variables [70]. At the same time, the module of the integral expression at the maximum point will behave as ε^{-l} , and the integration of the Gaussian integral that occurs when applying the Laplace method, as will be shown later, will lead to the appearance of the multiplier ε^l . This allows us to eliminate ε from the expression, after

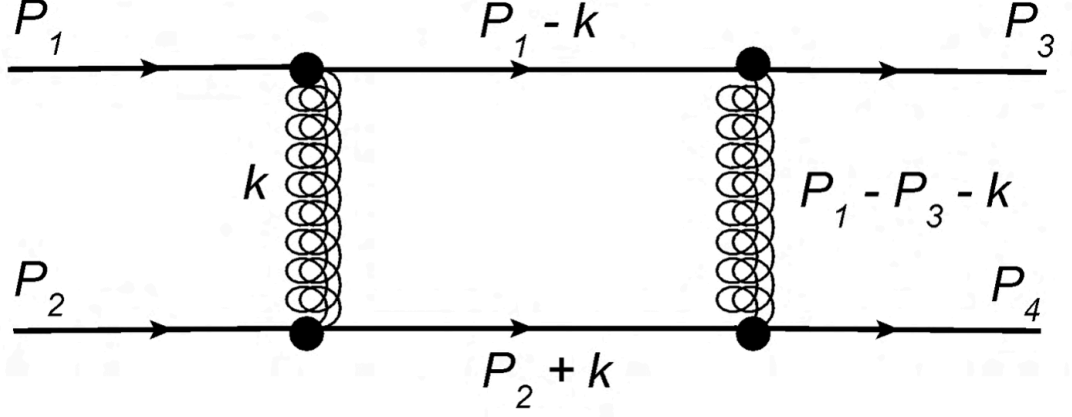


FIGURE 3.3: The simplest loop diagram of elastic proton scattering. P_1, P_2 - four momenta of incoming protons, P_3, P_4 - four momenta of outgoing protons, $k, P_1 - k, P_2 + k, P_1 - P_3 - k$ - four momenta of virtual particles. Double lines correspond to the bound state of gluons - glueballs.

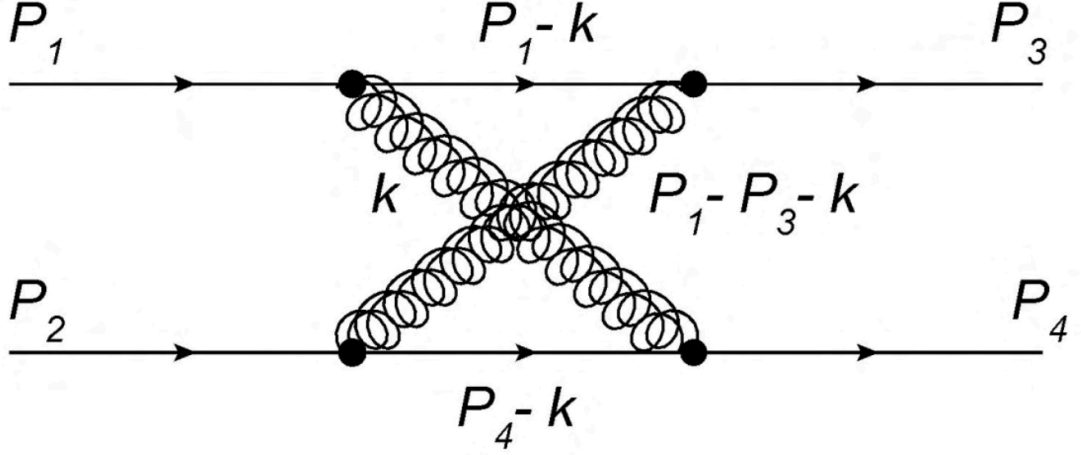


FIGURE 3.4: The simplest loop diagram of elastic proton scattering. P_1, P_2 - four momenta of incoming protons, P_3, P_4 - four momenta of outgoing protons, $k, P_1 - k, P_2 + k, P_1 - P_3 - k$ - four momenta of virtual particles. Double lines correspond to the bound state of gluons - glueballs.

which its approximation to zero is no longer a problem and can be done before integration, which can now be done using numerical methods.

This work considers the application of the presented idea for calculating the sum of the simplest proton elastic scattering diagram with one loop, shown in Fig. 3.3 and Fig. 3.4.

Therefore, the purpose of this work is to consider the calculations of

the model of the differential cross-section of elastic scattering within the limits of the multiparticle field method, based on taking into account the diagrams in Fig. 3.2, as well as the diagrams in Fig. 3.3, Fig. 3.4, and diagrams obtained from Fig. 3.3, Fig. 3.4 by permuting the lines of particles in the final state. The experimentally observed effects of the non-monotonicity of the dependence of the differential cross section of the elastic scattering of protons on the square of the transmitted four-momentum in our model are the consequences of spin effects.

3.3 Calculation of analytical expressions of R. Feynman's diagram with one loop using the Laplace method

3.3.1 Analysis of the expression for the simplest loop diagram

Let us take a look at the analytical expression for the simplest loop diagram showed in Fig. 3.3:

$$\begin{aligned}
A = & \frac{(ig)^4}{(2\pi)^6} (\bar{v}_{v_3}^+(P_4))_{s_4} \gamma_{s_4 s_2}^a (v_{v_2}^-(P_2))_{s_2} (\bar{v}_{v_4}^+(P_3))_{s_3} \gamma_{s_3 s_1}^b (v_{v_1}^-(P_1))_{s_1} \times \\
& \times \delta((P_3 + P_4) - (P_1 + P_2)) \int d^4k (k_a + 2P_{2a}) (2P_{1b} - k_b) \times \\
& \times \frac{1}{M_p^2 - (P_1 - k)^2 - i\varepsilon} \frac{1}{M_p^2 - (P_2 + k)^2 - i\varepsilon} \times \\
& \times \frac{1}{M_G^2 - k^2 - i\varepsilon} \frac{1}{M_G^2 - (P_1 - P_3 - k)^2 - i\varepsilon}
\end{aligned} \tag{3.1}$$

Here, g is the effective coupling constant, M_p and M_G are the proton and glueball masses, respectively, $(\bar{v}_{v_3}^+(P_4))_{s_4}, (v_{v_2}^-(P_2))_{s_2}, (\bar{v}_{v_4}^+(P_3))_{s_3},$

$(v_{v_1}^-(P_1))_{s_1}$ - solution of the Dirac equations, $\gamma_{s_4 s_2}^a, \gamma_{s_3 s_1}^b$ - elements of the Dirac matrices. All quantities are scaled to the proton mass. M_G and g are treated as fitted parameters. For ease of calculation, let us choose the centre-of-mass system of the particles. If we consider this system on the coordinate plane, then the coordinate axis z is directed along the vector $\vec{P}_1$, the coordinate axis x is chosen so that it is perpendicular to the axis z and lay in the plane of the vectors $\vec{P}_1$ and $\vec{P}_3$, and we choose the axis y so that it is perpendicular to the plane of the axes x and z .

To solve the problem formulated above, we need to calculate the tensor t_{ab} :

$$\begin{aligned}
t_{ab} = & \int d^4k (k_a + 2P_{2a}) (2P_{1b} - k_b) \times \\
& \times \frac{1}{M_p^2 - (P_1 - k)^2 - i\varepsilon} \frac{1}{M_p^2 - (P_2 + k)^2 - i\varepsilon} \times \\
& \times \frac{1}{M_G^2 - k^2 - i\varepsilon} \frac{1}{M_G^2 - (P_1 - P_3 - k)^2 - i\varepsilon}
\end{aligned} \tag{3.2}$$

Here, the indices a and b run through four values from 0 to 3 each. After calculating the tensor (3.2), the obtained result is averaged with the spin density matrix and we construct the differential cross section of the elastic scattering of protons by the square of the transmitted four-momentum based on the square of the modulus of the scattering amplitude, taking into account the contribution from the pole diagram.

To calculate the four-dimensional integral (3.2) we want to apply the Laplace method. To do this, we will apply the calculation method described in the introduction. First, let's take a small value of ε and see

which region makes the main contribution to the integral. If ε has a small value and is not equal to zero, then the maximum of the module of the integral expression will be where the largest number of real parts of the denominators turns to zero. Let's introduce the notation for the real parts of the denominators from the expression (3.2), i.e., this is the part of the denominator that does not contain ε :

$$\begin{aligned} z_1 &= M_p^2 - (P_1 - k)^2, z_2 = M_p^2 - (P_2 + k)^2 \\ z_3 &= M_G^2 - k^2, z_4 = M_G^2 - (P_1 - P_3 - k)^2 \end{aligned} \tag{3.3}$$

If we consider the square of the modulus of the denominator from the expression (3.2), then we have the square of the real part plus the square of ε , and if the real part of (3.3) is zero, and ε is not yet zero, then we will get not the divergence of the integral, but the maximum. Then, let one of (3.3) be zero, and this will select some subset on which this real part of the denominator is zero, and if we consider the integration already over this subset, then the main contribution to this integral will be made by the region in which the quality real parts are zero. As a result, the question arises, how many expressions with (3.3) can simultaneously rotate to zero. As was shown in the work of [75], the first two expressions z_1 and z_2 , which correspond to the horizontal lines of the P diagram, can rotate to zero simultaneously. Feynman (Fig.3.3), or two others z_3 and z_4 , which correspond to two vertical lines. That is, the horizontal and vertical lines cannot rotate to zero at the same time. Next, we will consider the calculations of the denominators in more detail.

3.3.2 Application of Laplace method to perform a limit transition $\varepsilon \rightarrow 0$

Next step in calculations is to consider the tensor 3.2 and introduce the notation for the numerator $f_{ab}(k^0, \vec{k}) = (k_a + 2(P_2)_a)(2(P_1)_b - k_b)$, where a and b take the values from 0 to 3. For the calculation, it is more convenient to choose the system of the center of mass, where $\vec{P}_2 = -\vec{P}_1$ and energy component of momenta is equal to $\frac{\sqrt{s}}{2}$. Let us rewrite the expression for tensor 3.2 in the form:

$$t_{ab} = \int_{-\infty}^{\infty} dk_0 \int d\vec{k} f_{ab}(k^0, \vec{k}) \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} - k^0\right)^2 + \left(\vec{P}_1 - \vec{k}\right)^2 - i\varepsilon} \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} + k^0\right)^2 + \left(\vec{P}_1 - \vec{k}\right)^2 - i\varepsilon} \frac{1}{M_G^2 - (k^0)^2 + \vec{k}^2 - i\varepsilon} \frac{1}{M_G^2 - (k^0)^2 + \left(\vec{P}_1 - \vec{P}_3 - \vec{k}\right)^2 - i\varepsilon} \quad (3.4)$$

Such a tensor 3.4 consists of 7 non-zero components which was calculated separately.

Since the main contribution to the tensor is made by denominators, it would be convenient to show a general example of solving its non-zero components. The denominators for all components will be the same, and the numerators will change. For further calculation of tensor 3.4, it is convenient to change a coordinate plane and use the one shown in Fig. 3.5.

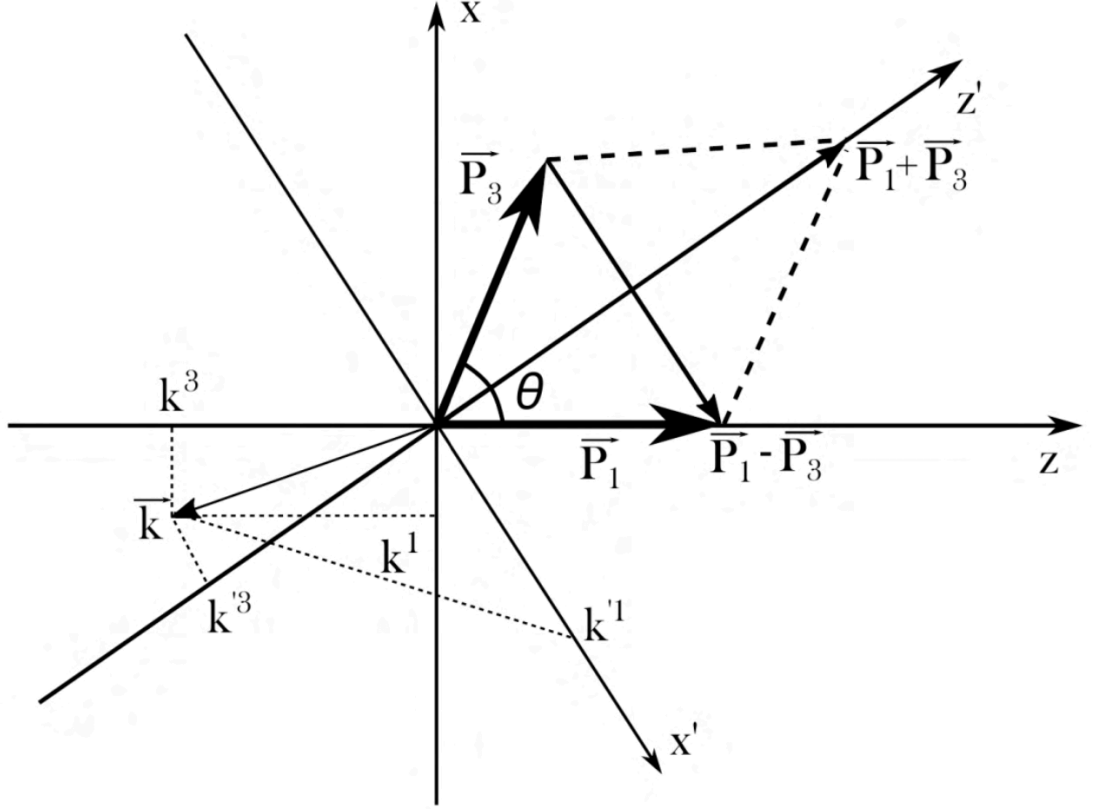


FIGURE 3.5: The location of the coordinates k^1 and k^3 of the vector $\vec{k}$ in the original coordinate system and k'^1 and k'^3 after rotation

The expression for the old coordinates in terms of the new ones after rotation is:

$$\begin{aligned} k^1 &= -k'^1 \cos\left(\frac{\theta}{2}\right) + k'^3 \sin\left(\frac{\theta}{2}\right), \\ k^3 &= k'^1 \sin\left(\frac{\theta}{2}\right) + k'^3 \cos\left(\frac{\theta}{2}\right) \end{aligned} \quad (3.5)$$

Also, for further calculation of 3.4, we will introduce the substitution of variables:

$$\frac{\vec{P}_1 - \vec{P}_3}{2} - \vec{k} = \vec{q}, \quad \vec{k} = \frac{\vec{P}_1 - \vec{P}_3}{2} - \vec{q}, \quad k^0 = q^0 \quad (3.6)$$

After a substitution, the tensor 3.4 takes the form:

$$\begin{aligned}
t_{ab} = & \int_{-\infty}^{\infty} dq_0 \int d\vec{q} f_{ab}(q^0, \vec{q}) \\
& \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + \left(\frac{\vec{P}_1 + \vec{P}_3}{2} + \vec{q}\right)^2 - i\varepsilon} \\
& \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + \left(\frac{\vec{P}_1 + \vec{P}_3}{2} + \vec{q}\right)^2 - i\varepsilon} \\
& \frac{1}{M_G^2 - (q^0)^2 + \left(\frac{\vec{P}_1 - \vec{P}_3}{2} - \vec{q}\right)^2 - i\varepsilon} \frac{1}{M_G^2 - (q^0)^2 + \left(\frac{\vec{P}_1 - \vec{P}_3}{2} + \vec{q}\right)^2 - i\varepsilon}
\end{aligned} \tag{3.7}$$

For further calculations it is more convenient to consider the expression for each of the denominators separately, since the expression for the tensor itself will increase after substitution and it is necessary to provide the full expression of the denominators. Let us write expressions for the denominators denoting their real parts as z_i , where $i = 1, \dots, 4$, according to the new coordinate plane and writing each of the components of our vectors separately:

$$\begin{aligned}
z_1 &= M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{|\vec{P}_1 + \vec{P}_3|}{2} + q^3\right)^2, \\
z_2 &= M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{|\vec{P}_1 + \vec{P}_3|}{2} + q^3\right)^2, \\
z_3 &= M_G^2 - (q^0)^2 + \left(\frac{|\vec{P}_1 - \vec{P}_3|}{2} - q^1\right)^2 + (q^2)^2 + (q^3)^2, \\
z_4 &= M_G^2 - (q^0)^2 + \left(\frac{|\vec{P}_1 - \vec{P}_3|}{2} + q^1\right)^2 + (q^2)^2 + (q^3)^2
\end{aligned} \tag{3.8}$$

Next step in the series of calculations is to express the vectors $\vec{P}_1$ and $\vec{P}_3$ through Mandelstam variable t . This step is needed because the cross section depends on the transmitted four-momentum t .

$$\begin{aligned} |\vec{P}_1 - \vec{P}_3| &= \sqrt{|t|}, \\ |\vec{P}_1 + \vec{P}_3| &= \sqrt{4P^2 - |t|}, \end{aligned} \tag{3.9}$$

where $P = |\vec{P}_1| = |\vec{P}_3|$.

The new expressions for the real parts of denominators 3.8 take the form:

$$\begin{aligned} z_1 &= M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0 \right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3 \right)^2, \\ z_2 &= M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0 \right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3 \right)^2, \\ z_3 &= M_G^2 - (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} - q^1 \right)^2 + (q^2)^2 + (q^3)^2, \\ z_4 &= M_G^2 - (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} + q^1 \right)^2 + (q^2)^2 + (q^3)^2 \end{aligned} \tag{3.10}$$

Introducing a re-definition $q^3 = \frac{\sqrt{4P^2 - |t|}}{2} + q^3$ for the third component of the vector $\vec{q}$ and passing to spherical coordinates (q, θ, φ) , we obtain expressions for its components

$$\begin{aligned} q^1 &= q \sin \theta \cos \varphi, \\ q^2 &= q \sin \theta \sin \varphi, \\ q^3 &= q \cos \varphi, \end{aligned} \tag{3.11}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. Now we can rewrite 3.10 as:

$$\begin{aligned}
z_1 &= M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0 \right)^2 + q^2, \\
z_2 &= M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0 \right)^2 + q^2, \\
z_3 &= M_G^2 - (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} - q \sin \theta \cos \varphi \right)^2 + (q \sin \theta \sin \varphi)^2 + \\
&\quad + \left(q \cos \varphi - \frac{\sqrt{4P^2 - |t|}}{2} \right)^2, \\
z_4 &= M_G^2 - (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} + q \sin \theta \cos \varphi \right)^2 + (q \sin \theta \sin \varphi)^2 + \\
&\quad + \left(q \cos \varphi - \frac{\sqrt{4P^2 - |t|}}{2} \right)^2
\end{aligned} \tag{3.12}$$

Taking into account 3.12 expression for the tensor 3.4 takes the form

$$\begin{aligned}
t_{ab} &= \int_{-\infty}^{\infty} dq_0 \int_0^{\infty} q^2 dq \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi f_{a,b}(q^0, q, \theta, \varphi) \\
&\quad \frac{1}{(z_1 - i\varepsilon)} \frac{1}{(z_2 - i\varepsilon)} \frac{1}{(z_3 - i\varepsilon)} \frac{1}{(z_4 - i\varepsilon)}
\end{aligned} \tag{3.13}$$

The idea of the Laplace method is that the integral can be nicely described by the corresponding Gaussian integral if the integrand of it has a single maximum point in the integration domain [71, 72].

As was shown in [75], either the first two real parts of the first denominators z_1 and z_2 , which correspond to the horizontal lines of the R. Feynman diagram (Fig. 3.3), or the other two real parts of the other two denominators z_3 and z_4 , which correspond to the two vertical lines, can simultaneously turn into zero. That means that the real parts of the

denominators corresponding to the horizontal and vertical lines cannot turn to zero simultaneously.

3.3.2.1 Calculation of the maximum contribution of the first two denominators

Next step is to find the regions of the integration domain where either the first pair or second pair of denominators turn to zero and calculate the contributions of those regions to the main integral. Thus, let us consider the system of equations of the first two denominators z_1 and z_2 from expression 3.13 and set the real parts of them equal to zero. The remaining $1/\varepsilon^2$ will give the maximum contribution of the integrand

$$\begin{cases} M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + q^2 = 0, \\ M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + q^2 = 0 \end{cases} \quad (3.14)$$

The solution $q^0 = 0$ and $q = P$ satisfies the system of equations 3.14, which means that from the four-dimensional integration domain we have selected a two-dimensional subset of spherical shape, which is defined by the simple equations $q^0 = 0$ and $q = P$. On this subset the absolute value of the first two denominators has a single minimum. Let us represent the first two denominators as the exponent of the logarithm

of these denominators:

$$\begin{aligned}
& \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + q^2 - i\varepsilon} \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + q^2 - i\varepsilon} = \\
& = \exp \left[-\ln \left(M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + q^2 - i\varepsilon \right) - \right. \\
& \quad \left. - \ln \left(M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + q^2 - i\varepsilon \right) \right] = \\
& = -\frac{1}{\varepsilon^2} \exp \left[-\ln \left(1 - \frac{M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + q^2}{i\varepsilon} \right) - \ln \left(1 - \frac{M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + q^2}{i\varepsilon} \right) \right]
\end{aligned} \tag{3.15}$$

Considering the Taylor expansion of 3.15 to second-order derivatives with a new variable substitution $q = x + P$, we obtain:

$$\begin{aligned}
& \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + (x + P)^2 - i\varepsilon} \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + (x + P)^2 - i\varepsilon} \approx \\
& -\frac{1}{\varepsilon^2} \exp \left[\frac{s(q^0)^2 + 4P^2x^2}{-\varepsilon^2} + \frac{-2(q^0)^2 + 4Px + 2x^2}{i\varepsilon} \right]
\end{aligned} \tag{3.16}$$

It is convenient to change the variable $q = x + P$ because together with q^0 , they are the offset from the point ($q^0 = 0, q = P$) where, as it was mentioned above, the absolute value of the first two denominators take a minimum value. It follows that the absolute value of the exponent in 3.16 has a maximum in this point.

The approximation for the tensor 3.13 itself is

$$t_{ab} \approx -\frac{1}{\varepsilon^2} \int_{-\infty}^{\infty} dq_0 \int_{-P}^{\infty} (P+x)^2 dx \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi \left[f_{a,b}(q^0, q, \theta, \varphi) \right. \\ \left. \times \exp \left(\frac{s(q^0)^2 + 4P^2 x^2}{-\varepsilon^2} + \frac{-2(q^0)^2 + 4Px + 2x^2}{i\varepsilon} \right) \frac{1}{(z_3 - i\varepsilon)} \frac{1}{(z_4 - i\varepsilon)} \right], \quad (3.17)$$

where the same substitution $q = x + P$ is performed for f_{ab} , z_3 and z_4 .

Since all the calculations are performed based on the fact that the absolute value of the exponent in 3.16 has a maximum in specific point, it is better to analyze the integrand in 3.17.

Let us denote the integrand as $a_{\varepsilon}(q^0, x, \theta, \varphi)$. The exponent we are interested in depends on variables q^0 and x . The width of the peak in the maximum point tends to zero when $\varepsilon \rightarrow 0$. Since the the part with z_3 and z_4 denominators in 3.17 does not have singularities at the point $(q^0 = 0, q = P)$, to cut the integration domain we can use the following exponential suppression of the integrand

$$\int_{-\infty}^{\infty} dq^0 \int_{-P}^{\infty} dx \rightarrow \int_{-q_{cut}^0}^{q_{cut}^0} dq^0 \int_{-x_{cut}}^{x_{cut}} dx, \quad (3.18)$$

where $q_{cut}^0 > 0$ and $0 < x_{cut} < P$.

After applying of the suppression it was found that the function $a_{\varepsilon}(q^0 = 0, x = 0, \theta, \varphi)$ has a maximum when θ and φ are equal to zero [70]. Considering the restriction of $a_{\varepsilon}(q^0, x, \theta, \varphi)$ to $a_{\varepsilon}(q^0, x, \theta = 0, \varphi = 0)$, which is denoted as $a_{\varepsilon}(q^0, x)$ we can plot the absolute value in the $q^0 x$ plane, as it showed in Fig. 3.6a.

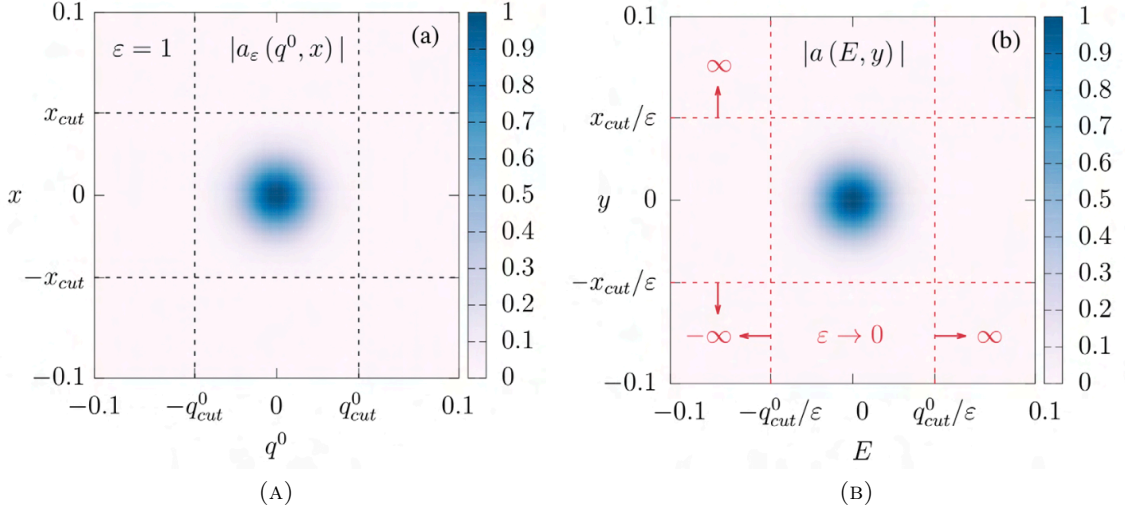


FIGURE 3.6: Color map plot of $|a_\varepsilon(q^0, x)|$ in q^0x plane (A), and $|a(E, y)|$ in Ey plane (B) for $\varepsilon = 1$. The values of parameters at $\sqrt{s} = 44.6$ GeV and $M_p = 0.938$ GeV.

In such a way we are considering the contribution to the integral 3.13 from the first pair of denominators z_1 and z_2 . It is worth to mention that the actual value of the contribution to the integral does not depend on the selection of cuts q_{cut}^0 and x_{cut} . To show it, we are going to make a substitution

$$x = \varepsilon y, \quad q^0 = \varepsilon E. \quad (3.19)$$

Performing 3.19 on 3.18 we obtain

$$\int_{-\infty}^{\infty} dq^0 \int_{-P}^{\infty} dx \rightarrow \varepsilon^2 \int_{-q_{cut}^0/\varepsilon}^{q_{cut}^0/\varepsilon} dE \int_{-x_{cut}/\varepsilon}^{x_{cut}/\varepsilon} dy \quad (3.20)$$

After such substitution which cancels all the ε in denominators z_1 and z_2 , tensor 3.13 can be rewritten as

$$t_{ab} \approx - \int_{-\infty}^{\infty} dE \int_{-P/\varepsilon}^{\infty} (P + \varepsilon y)^2 dy \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi \left[f_{a,b}(q^0, q, \theta, \varphi) \right. \\ \left. \times \exp(-sE^2 - 4P^2y^2 + 2i\varepsilon E^2 - 4iPy - 2i\varepsilon y^2) \frac{1}{(z_3 - i\varepsilon)} \frac{1}{(z_4 - i\varepsilon)} \right] \quad (3.21)$$

Since now we do not face with a problem of bypassing the poles because there is no ε in the first pair of denominators, it allows us to finally pass to the limit when $\varepsilon \rightarrow 0$. It is also seen that since both q_{cut}^0/ε and x_{cut}/ε tend to infinity when $\varepsilon \rightarrow 0$ (Fig. 3.6b), while the width of the peak of $|a(E, y)|$ remains unchanged, the resulting integration limits will not depend on the particular selection of q_{cut}^0 and x_{cut} .

After passing to the limit when $\varepsilon \rightarrow 0$, tensor 3.21 takes the form

$$t_{ab} \approx -P^2 \int_{-\infty}^{\infty} dE \int_{-P/\varepsilon}^{\infty} dy \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi \left[f_{a,b}(\theta, \varphi) \right. \\ \left. \times \exp(-sE^2 - 4P^2y^2 - 4iPy) \frac{1}{(z_3 z_4)} \right], \quad (3.22)$$

where $f_{a,b}(\theta, \varphi)$ denotes $f_{a,b}(q^0 = 0, x = 0, \theta, \varphi)$. Integration over the variable E is reduced to Poisson integral

$$\int_{-\infty}^{\infty} dE \exp(-sE^2) = \frac{\sqrt{\pi}}{\sqrt{s}} \quad (3.23)$$

Similarly, we can reduce integral over y to Poisson integral:

$$\begin{aligned}
& \int_{-\infty}^{\infty} dy \exp(-4P^2 y^2 - 4iPy) = \int_{-\infty}^{\infty} dy \exp(-4P^2 y^2 - 4iPy - 1 + 1) = \\
& = \int_{-\infty}^{\infty} dy \exp\left(-\left(4P^2 y^2 + 4iPy - 1 + 1\right)\right) = \\
& = \int_{-\infty}^{\infty} dy \exp\left(-\left((2Py + i)^2 + 1\right)\right) = \\
& = \frac{1}{e} \int_{-\infty}^{\infty} dy \exp\left(-(2Py + i)^2\right) = [2Py + i = z] = \\
& = \frac{1}{2eP} \int_{-\infty+i}^{\infty+i} dz \exp(-z^2) = \frac{\sqrt{\pi}}{2eP},
\end{aligned} \tag{3.24}$$

where e is the Euler's number (i.e., the base of the natural logarithm).

Substituting solutions [3.23](#) and [3.24](#) in [3.22](#) we obtain

$$\begin{aligned}
t_{ab} \approx & -\frac{\pi P}{2e\sqrt{s}} \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\varphi \left[f_{a,b}(\theta, \varphi) \right. \\
& \left. \times \frac{1}{\left(M_G^2 + 2P^2 - P \cos \theta \sqrt{4P^2 - |t|}\right)^2 - P^2 |t| \sin^2 \theta \cos^2 \varphi} \right].
\end{aligned} \tag{3.25}$$

Finally, only the two-dimensional integral over θ and φ for each component of t_{ab} remains. Furthermore, the integral over φ can be calculated analytically for each pair of indices (a, b) although these calculations are quite long but yet straightforward. So, we are ending up with a one dimensional integral over θ which can be calculated numerically.

3.3.2.2 Divergency problem in the second pair of tensor denominators

Since the main contribution from first pair of denominators was found, let us change our focus to the second pair of denominators z_3 and z_4 in 3.4:

$$\begin{aligned} \frac{1}{(z_3 - i\varepsilon)} \frac{1}{(z_4 - i\varepsilon)} &= \frac{1}{M_G^2 - (k^0)^2 + \vec{k}^2 - i\varepsilon} \\ &\times \frac{1}{M_G^2 - (k^0)^2 + \left(\vec{P}_1 - \vec{P}_3 - \vec{k}\right)^2 - i\varepsilon} \end{aligned} \quad (3.26)$$

In these denominators the divergence arises if we tend to infinity simultaneously both the energy component k^0 of four-momentum k and the length of its spatial part $|\vec{k}|$. The difference of their squares $(k^0)^2 - |\vec{k}|^2$ may still be finite, but their sum $(k^0)^2 + |\vec{k}|^2$ takes infinite values when either k^0 or $|\vec{k}|$ tends to infinity. It is worth to specify that the first pair of denominators contains k^0 in the first power. It provides the decreasing of the integrand when $k^0 \rightarrow 0$. Nevertheless, in the center of mass system, in the second pair of denominators 3.26 the energy component $P_1^0 - P_3^0$ is equal to zero which leads us to the second power of k^0 and they will contain the finite value of $(k^0)^2 - |\vec{k}|^2$. In this case the whole integrand goes to zero due to the first power of k^0 in first pair of denominators.

If we look at the first tensor 3.2 component t_{00} , in the numerator we have $f_{ab}(k^0, \vec{k}) = s - (k^0)^2$, where s is the Mandelstam invariant which tends to infinity just like the denominator of the integrand. In this case the whole integral diverges. But it is possible to show that this divergence is only "delusive" as it will be shown in further discussion.

Considering the tensor after variables substitution 3.7, we can show that it is possible to turn the difference $(q^0)^2 - |\vec{q}|^2$ into a sum and then there will be no problem with divergence. It is possible to make it by the Wick rotation, by transforming the integration contour in the complex plane from the real axis to the imaginary one. To do that we firstly need to take a look at the integrand poles since the possibility of this transformation depends on the location of the poles.

Let us rewrite the expression 3.7 in the form

$$t_{ab} = \int_{-\infty}^{\infty} dq^0 \int_{-\infty}^{\infty} d\vec{q} f_{ab}(q^0, \vec{q}) \frac{1}{q^0 - \left(-\frac{\sqrt{s}}{2} + \eta\right)} \frac{1}{q^0 - \left(-\frac{\sqrt{s}}{2} - \eta\right)} \frac{1}{q^0 - \left(\frac{\sqrt{s}}{2} + \eta\right)} \frac{1}{q^0 - \left(\frac{\sqrt{s}}{2} - \eta\right)} \frac{1}{(z_3 - i\varepsilon)} \frac{1}{(z_4 - i\varepsilon)}, \quad (3.27)$$

where $\eta = \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|^2}}{2} + q^3\right)^2} - i\varepsilon$.

Next step is to analyze the placement of quantities η and $-\eta$ on the complex plane q^0 . The location of the poles in the complex plane q^0 depends on the values of the variables q^1, q^2, q^3 . In other words in different subsets of the integration domain with respect to q^1, q^2 , and q^3 , we have different location of the poles. Also, the important question is which of two conditions

$$\eta|_{\varepsilon=0} > \frac{\sqrt{s}}{2} \quad \text{or} \quad \eta|_{\varepsilon=0} < \frac{\sqrt{s}}{2} \quad (3.28)$$

is implemented on the considered subset. It is important because the distribution of the poles over the quarters of the complex plane q^0 depends

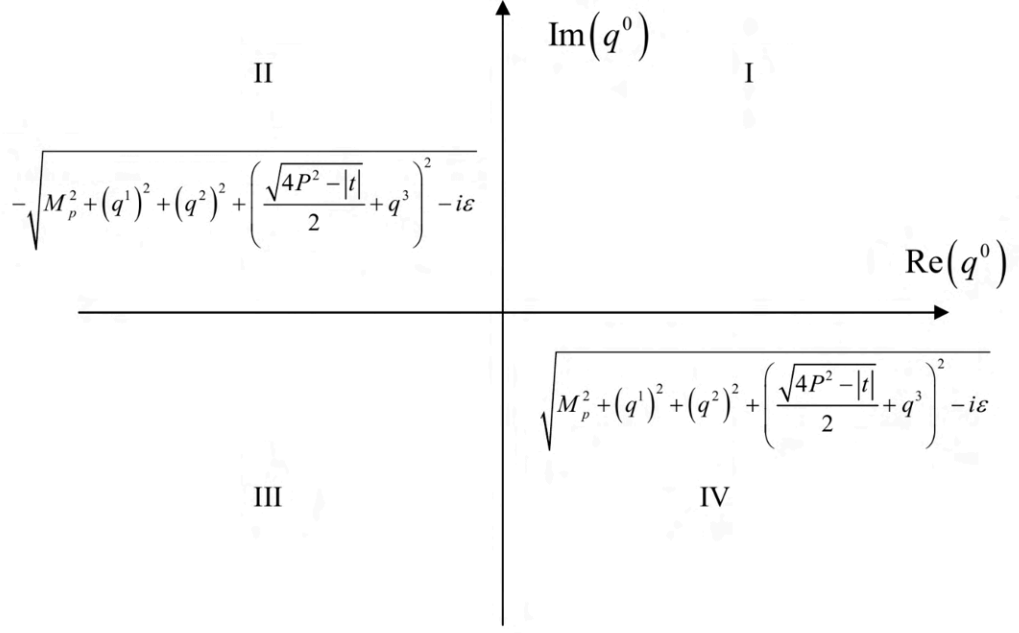


FIGURE 3.7: Consideration of quantities η and $-\eta$ on the complex plane q^0

on the realization of first or second condition. In case the condition $\eta|_{\varepsilon=0} < \frac{\sqrt{s}}{2}$ holds, the poles of the integrand 3.27 are located in all quarters of the complex plane q^0 . Therefore we cannot apply the Wick rotation, because during the rotation of integration path we will cross the poles.

In case of the first condition $\eta|_{\varepsilon=0} > \frac{\sqrt{s}}{2}$, the poles are located in the second and fourth quadrants (Fig. 3.7), so we can turn the contour, i.e. make a Wick rotation. Integration over the variable q^0 will change to integration over the variable iq^0 and $(q^0)^2 - |\vec{q}|^2$ will be replaced by $-((q^0)^2 - |\vec{q}|^2)$. In this sum if one of the components goes to infinity, the whole sum will also go to infinity. In this case, the problem of divergency is solved. The integrand will go to zero at the values of the integration variables as $((q^0)^2 - |\vec{q}|^2)^{-4}$, which ensures convergence of the integral regardless of the different options of the expression f_{ab} in the numerator for different components of the tensor t_{ab} .

The possibility of applying the Wick rotation arises only in the high $|\vec{q}|$ values, where the problem of divergence appears. The rest of the integration domain is not causing a divergence because when $\varepsilon \neq 0$ there are no singularities of the integral and this domain is finite [73]. That means, if we select a specific region and apply corresponding Wick rotation to it, we obtain convergent integral. This was the main reason of calling this divergence of the integral "delusive" earlier.

3.3.2.3 Calculation of the maximum contribution of the second two denominators with the Wick rotation

Implementing the considerations given in the previous section, we will split the integral in expression 3.7 into 2 integrals. The integral over a finite domain in which it is impossible to make a Wick rotation is denoted by:

$$\begin{aligned}
t_{ab}^< &= \int_{-\infty}^{\infty} dq_0 \int_{|\vec{q}| < cP} d\vec{q} f_{ab}(q^0, \vec{q}) \\
&\frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{|\vec{P}_1 + \vec{P}_3|}{2} + q^3\right)^2 - i\varepsilon} \\
&\frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{|\vec{P}_1 + \vec{P}_3|}{2} + q^3\right)^2 - i\varepsilon} \\
&\frac{1}{M_G^2 - (q^0)^2 + \left(\frac{|\vec{P}_1 - \vec{P}_3|}{2} - q^1\right)^2 + (q^2)^2 + (q^3)^2 - i\varepsilon} \\
&\frac{1}{M_G^2 - (q^0)^2 + \left(\frac{|\vec{P}_1 - \vec{P}_3|}{2} + q^1\right)^2 + (q^2)^2 + (q^3)^2 - i\varepsilon}
\end{aligned} \tag{3.29}$$

Here, the number c can be given different values greater than 1. It will be seen that the result of integration does not depend on the choice of this number. The value of c must be greater than 1, because if $c = 1$ under the condition $|\vec{q}| < P$ the poles in the integrand 3.27 will be placed on the integration axis and we obtain a divergent integral. The contribution to the tensor t_{ab} in the region where the Wick rotation is possible is denoted as follows:

$$\begin{aligned}
t_{ab}^> &= \int_{-\infty}^{\infty} dq_0 \int_{|\vec{q}| > cP} d\vec{q} f_{ab}(q^0, \vec{q}) \\
&\frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} - q^0\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{|\vec{P}_1 + \vec{P}_3|}{2} + q^3\right)^2 - i\varepsilon} \\
&\frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} + q^0\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{|\vec{P}_1 + \vec{P}_3|}{2} + q^3\right)^2 - i\varepsilon} \quad (3.30) \\
&\frac{1}{M_G^2 - (q^0)^2 + \left(\frac{|\vec{P}_1 - \vec{P}_3|}{2} - q^1\right)^2 + (q^2)^2 + (q^3)^2 - i\varepsilon} \\
&\frac{1}{M_G^2 - (q^0)^2 + \left(\frac{|\vec{P}_1 - \vec{P}_3|}{2} + q^1\right)^2 + (q^2)^2 + (q^3)^2 - i\varepsilon}
\end{aligned}$$

where real parts of denominators in second, third, fourth and fifth lines are z_1 , z_2 , z_3 and z_4 respectively. Same notations were applied for 3.29.

Expressing $\vec{P}_1$ and $\vec{P}_3$ through t 3.9, we consider the system of equations for the second two denominators by setting the real parts equal to zero and then finding the conditional second maxima:

$$\begin{cases} M_G^2 - (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} - q^1\right)^2 + (q^2)^2 + (q^3)^2 = 0, \\ M_G^2 - (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} + q^1\right)^2 + (q^2)^2 + (q^3)^2 = 0 \end{cases} \quad (3.31)$$

Making a notation $\chi = \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2}$ we can clearly see that the solution $q^1 = 0$, $q^0 = \pm\chi$ satisfies this system of equations. It is worth to highlight that χ takes both positive and negative values but we will use only the positive solution in the following calculations, since the integrand is an even function of q^0 .

Introducing the change of variables in expression [3.34](#)

$$\begin{aligned} q^0 &= \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2} + \varepsilon E, \\ q^1 &= \varepsilon x \end{aligned} \tag{3.32}$$

and representing the second pair of denominators from [3.29](#) in exponential form

$$\begin{aligned} \frac{1}{z_3 - i\varepsilon} &= \frac{1}{\sqrt{(z_3)^2 + \varepsilon^2}} \exp \left(i \arccos \frac{z_3}{\sqrt{(z_3)^2 + \varepsilon^2}} \right), \\ \frac{1}{z_4 - i\varepsilon} &= \frac{1}{\sqrt{(z_4)^2 + \varepsilon^2}} \exp \left(i \arccos \frac{z_4}{\sqrt{(z_4)^2 + \varepsilon^2}} \right) \end{aligned} \tag{3.33}$$

we can consider the integral over a finite domain in which it is impossible to make a Wick rotation ([3.29](#)). The new expression for tensor t_{ab} after applying [3.33](#) and making a substitution [3.9](#) takes the form

$$\begin{aligned} t_{ab}^< &= 2 \int_{-\infty}^{\infty} dq_0 \int_{-cP}^{cP} dq_1 \int_{|\vec{q}| > cP} dq^1 \int_{(q^2)^2 + (q^3)^2 < c^2 P^2 - (q^1)^2} dq^2 dq^3 f_{ab}(q^0, \vec{q}) \\ &\frac{1}{z_1 - i\varepsilon} \frac{1}{z_2 - i\varepsilon} \frac{1}{\sqrt{(z_3)^2 + \varepsilon^2}} \exp \left(i \arccos \frac{z_3}{\sqrt{(z_3)^2 + \varepsilon^2}} \right) \\ &\frac{1}{\sqrt{(z_4)^2 + \varepsilon^2}} \exp \left(i \arccos \frac{z_4}{\sqrt{(z_4)^2 + \varepsilon^2}} \right) \end{aligned} \tag{3.34}$$

where z_1, z_2, z_3 and z_4 are expressions from the formula 3.10.

After the substitution 3.32, the problem with $\varepsilon \rightarrow 0$ disappears because now we can remove ε^2 in the numerator and denominator and cancel them.

Passing to the limit when $\varepsilon \rightarrow 0$, the new expression for tensor t_{ab} (3.34) becomes

$$\begin{aligned}
t_{ab}^< = & 2 \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dx \int_{D^<} dq^2 dq^3 f_{ab}(q^2, q^3) \frac{1}{z_1 z_2} \\
& \frac{1}{\sqrt{(-2E\chi - \sqrt{|t|x})^2 + 1}} \exp \left(i \arccos \left(\frac{-2E\chi - \sqrt{|t|x}}{\sqrt{(-2E\chi - \sqrt{|t|x})^2 + 1}} \right) \right) \\
& \frac{1}{\sqrt{(-2E\chi + \sqrt{|t|x})^2 + 1}} \exp \left(i \arccos \left(\frac{-2E\chi + \sqrt{|t|x}}{\sqrt{(-2E\chi + \sqrt{|t|x})^2 + 1}} \right) \right)
\end{aligned} \tag{3.35}$$

where $D^< = \left\{ (q^2, q^3) \in R^2 \mid (q^2)^2 + (q^3)^2 < c^2 P^2 \right\}$ and $f_{ab}(q^2, q^3)$ denotes $f_{ab}(q^0 = \chi, q^1 = 0, q^2, q^3)$.

Next, we represent the square roots in the third and fourth denominators in the form of the exponent of the logarithm and expand the exponent in a Taylor series in the vicinity of zero values of E and x , limiting ourselves to the second order contributions according to the Laplace

method [65]:

$$\begin{aligned} \exp \left(\ln \left(\frac{1}{\sqrt{(-2E\chi - \sqrt{|t|x})^2 + 1}} \times \frac{1}{\sqrt{(-2E\chi + \sqrt{|t|x})^2 + 1}} \right) \right) &\approx \\ \approx \exp(-4E^2\chi^2 - |t|x^2) & \end{aligned} \quad (3.36)$$

We decompose the other two terms in these denominators similar to 3.36:

$$\begin{aligned} \exp \left(i \arccos \left(\frac{-2E\chi - \sqrt{|t|x}}{\sqrt{(-2E\chi - \sqrt{|t|x})^2 + 1}} \right) \right) &\times \\ \times \exp \left(i \arccos \left(\frac{-2E\chi + \sqrt{|t|x}}{\sqrt{(-2E\chi + \sqrt{|t|x})^2 + 1}} \right) \right) &= \\ = \exp \left(i \left(\frac{\pi}{2} + 2E\chi + \sqrt{|t|x} \right) \right) &\times \exp \left(i \left(\frac{\pi}{2} + 2E\chi - \sqrt{|t|x} \right) \right) \approx \\ \approx -\exp(-4iE\chi) & \end{aligned} \quad (3.37)$$

Substituting obtained expressions 3.36 and 3.37 into 3.35, the integration over variables E and x is now reduced to the Poisson integrals

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-|t|x^2) &= \frac{\sqrt{\pi}}{\sqrt{|t|}} \\ \int_{-\infty}^{\infty} \exp(-4E^2\chi^2 + 4iE\chi) &= \frac{\sqrt{\pi}}{2e\chi} \end{aligned} \quad (3.38)$$

Taking into account transformations 3.38 and writing the full expression instead of the notation for χ , the tensor 3.35 takes the following form

$$t_{ab}^< \approx -\frac{\pi}{e\sqrt{|t|}} \int_{D^<} dq^2 dq^3 \frac{f_{ab}(q^2, q^3)}{\sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2}} \times \frac{1}{\left(M_G^2 + \frac{|t|}{2} - q^3 \sqrt{4P^2 - |t|}\right)^2 - s \left(M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2\right)} \quad (3.39)$$

For further calculations it is convenient to use the polar coordinates

$$\begin{aligned} q^2 &= q \cos(\alpha), \\ q^3 &= q \sin(\alpha) \end{aligned} \quad (3.40)$$

The expression 3.39 become

$$t_{ab}^< = -\frac{\pi}{e\sqrt{|t|}} \int_0^{cP} q dq \frac{f_{ab}(q, \alpha)}{\sqrt{M_G^2 + \frac{|t|}{4} + q^2}} \times \int_0^{2\pi} d\alpha \frac{1}{\left(M_G^2 + \frac{|t|}{2} - q \sin(\alpha) \sqrt{4P^2 - |t|}\right)^2 - s \left(M_G^2 + \frac{|t|}{4} + q^2\right)} \quad (3.41)$$

We reduced the expression for the integral 3.29 to a two-dimensional integral in the variables q and α , which was calculated numerically for each component of the tensor t_{ab} separately.

Now we are going back to the integral 3.30 and applying the Wick rotation. On the new path of integration we are making a substitution

$q^0 \rightarrow iq^0$ in the integrand and 3.30 will take the form

$$\begin{aligned}
t_{ab}^> = & i \int_{-\infty}^{\infty} dq^0 \int_{-\infty}^{\infty} dq^2 \int_{D^>} dq^1 dq^3 \left[f_{ab}(q^0, q^1, q^2, q^3) \times \right. \\
& \times \frac{1}{M_p^2 + \left(q^0 - i\frac{\sqrt{s}}{2}\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3\right)^2 - i\varepsilon} \times \\
& \times \frac{1}{M_p^2 + \left(q^0 + i\frac{\sqrt{s}}{2}\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3\right)^2 - i\varepsilon} \times \\
& \times \frac{1}{M_G^2 + (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} - q^1\right)^2 + (q^2)^2 + (q^3)^2 - i\varepsilon} \times \\
& \left. \times \frac{1}{M_G^2 + (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} + q^1\right)^2 + (q^2)^2 + (q^3)^2 - i\varepsilon} \right] \quad (3.42)
\end{aligned}$$

where $D^> = \{(q^1, q^3) \in R^2 \mid (q^1)^2 + (q^3)^2 > c^2 P^2\}$.

Let us remind you that in the previous expressions for the first two denominators and for the second ones until the Wick rotation, we use a common notation in the numerator f_{ab} , which includes different numerators from different components of the tensor $t_a b$. We calculated them separately. It is better to open the brackets with q^0 and q^1 components in our denominators, so we can write the product of each pair $z_1 \cdot z_2$ and $z_3 \cdot z_4$ as the difference of squares.

Since this integral is convergent, we can pass to the limit $\varepsilon \rightarrow 0$ and rewrite 3.42 as

$$\begin{aligned}
t_{ab}^> = & i \int_{-\infty}^{\infty} dq^0 \int_{-\infty}^{\infty} dq^2 \int_{D^>} dq^1 dq^3 \left[f_{ab}(q^0, q^1, q^2, q^3) \times \right. \\
& \times \frac{1}{\left(M_p^2 + (q^0)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3 \right)^2 - \frac{s}{4} \right)^2 + q^0 s} \times \\
& \left. \times \frac{1}{\left(M_G^2 + (q^0)^2 + (q^1)^2 + (q^2)^2 + (q^3)^2 + \frac{|t|}{4} \right)^2 - q^1 |t|} \right]
\end{aligned} \tag{3.43}$$

It is clear that the maximum of this function is $q^0 = 0, q^2 = 0$, since the function itself is even with regard to q^0 and q^2 . This means that increasing or decreasing of the absolute values of q^0 and q^2 will lead to increasing or decreasing denominators and therefore decreasing or increasing the fraction itself. That is why in 3.42 we separate the integration with respect to q^0, q^2 and q^1, q^3 . To simplify further calculations it is convenient to introduce the following polar coordinates

$$q^1 = q \cos(\alpha), \quad q^3 = q \sin(\alpha). \tag{3.44}$$

and rewrite expression 3.42:

$$\begin{aligned}
t_{ab}^> = & i \int_{cP}^{\infty} q dq \int_0^{2\pi} d\alpha \int_{-\infty}^{\infty} dq^0 \int_{-\infty}^{\infty} dq^2 f_{ab}(q, \alpha, q^0, q^2) \times \\
& \times \frac{1}{M_p^2 + \left(q^0 - i\frac{\sqrt{s}}{2}\right)^2 + q^2 \cos^2 \alpha + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q \sin \alpha\right)^2} \times \\
& \times \frac{1}{M_p^2 + \left(q^0 + i\frac{\sqrt{s}}{2}\right)^2 + q^2 \cos^2 \alpha + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q \sin \alpha\right)^2} \times \\
& \times \frac{1}{M_G^2 + (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} - q \cos \alpha\right)^2 + (q^2)^2 + q^2 \sin^2 \alpha} \times \\
& \times \frac{1}{M_G^2 + (q^0)^2 + \left(\frac{\sqrt{|t|}}{2} + q \cos \alpha\right)^2 + (q^2)^2 + q^2 \sin^2 \alpha}
\end{aligned} \tag{3.45}$$

It is worth reminding you that now the integral is convergent, and integration is performed along the imaginary axis. Since 3.45 contains integration over the variable q in the limits from cP to infinity, to make our limits finite it is convenient to proceed to the next substitution $q =$

$1/x$. New expression for 3.45 takes the form

$$\begin{aligned}
t_{ab}^> = & i \int_0^{\frac{1}{cP}} x^5 dq \int_0^{2\pi} d\alpha \int_{-\infty}^{\infty} dq^0 \int_{-\infty}^{\infty} dq^2 f_{ab}(x, \alpha, q^0, q^2) \times \\
& \times \frac{1}{x^2 \left(M_p^2 + \left(q^0 - i\frac{\sqrt{s}}{2} \right)^2 + (q^2)^2 \right) + \cos^2 \alpha + \left(x \frac{\sqrt{4P^2 - |t|}}{2} + \sin \alpha \right)^2} \times \\
& \times \frac{1}{x^2 \left(M_p^2 + \left(q^0 + i\frac{\sqrt{s}}{2} \right)^2 + (q^2)^2 \right) + \cos^2 \alpha + \left(x \frac{\sqrt{4P^2 - |t|}}{2} + \sin \alpha \right)^2} \times \\
& \times \frac{1}{x^2 \left(M_G^2 + (q^0)^2 + (q^2)^2 \right) + \left(x \frac{\sqrt{|t|}}{2} - \cos \alpha \right)^2 + \sin^2 \alpha} \times \\
& \times \frac{1}{x^2 \left(M_G^2 + (q^0)^2 + (q^2)^2 \right) + \left(x \frac{\sqrt{|t|}}{2} + \cos \alpha \right)^2 + \sin^2 \alpha}
\end{aligned} \tag{3.46}$$

Next step is to make a substitution for $q^0 \rightarrow E = xq^0$ and $q^2 \rightarrow y = xq^2$ and open all brackets in the denominators:

$$\begin{aligned}
t_{ab}^> = & i \int_0^{\frac{1}{cP}} x^5 dq \int_0^{2\pi} d\alpha \int_{-\infty}^{\infty} dq^0 \int_{-\infty}^{\infty} dq^2 f_{ab}(x, \alpha, q^0, q^2) \times \\
& \times \frac{1}{E^2 - ixE\sqrt{s} + y^2 + 1 - x^2 \frac{|t|}{4} + x\sqrt{4P^2 - |t|} \sin \alpha} \times \\
& \times \frac{1}{E^2 + ixE\sqrt{s} + y^2 + 1 - x^2 \frac{|t|}{4} + x\sqrt{4P^2 - |t|} \sin \alpha} \times \\
& \times \frac{1}{x^2 M_G^2 + E^2 + y^2 + x^2 \frac{|t|}{4} - x\sqrt{|t|} \cos \alpha + 1} \times \\
& \times \frac{1}{x^2 M_G^2 + E^2 + y^2 + x^2 \frac{|t|}{4} + x\sqrt{|t|} \cos \alpha + 1}
\end{aligned} \tag{3.47}$$

Because the expression for the tensor 3.47 becomes very large, let us introduce following notation:

$$\begin{aligned} w_1(x, \alpha) &= 1 - x^2 \frac{|t|}{4} + x \sqrt{4P^2 - |t|} \sin \alpha, \\ w_2(x, \alpha) &= x^2 M_G^2 + x^2 \frac{|t|}{4} - x \sqrt{|t|} \cos \alpha + 1, \\ w_3(x, \alpha) &= x^2 M_G^2 + x^2 \frac{|t|}{4} + x \sqrt{|t|} \cos \alpha + 1 \end{aligned} \quad (3.48)$$

After taking these expressions out of parentheses in all denominators, we are making an additional notation:

$$\begin{aligned} W_1(x, \alpha) &= \frac{2}{w_1(x, \alpha)} + \frac{1}{w_2(x, \alpha)} + \frac{1}{w_3(x, \alpha)}, \\ W_2(x, \alpha) &= \frac{x^2 s}{(w_1(x, \alpha))^2} \end{aligned} \quad (3.49)$$

Taking into account 3.48 and 3.49, we obtain the new expression for tensor 3.47

$$\begin{aligned} t_{ab}^> &= i \int_0^{\frac{1}{cP}} x dq \int_0^{2\pi} d\alpha \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dy f_{ab}(x, E, \alpha, y) \times \\ &\times \frac{1}{(w_1(x, \alpha))^2 w_2(x, \alpha) w_3(x, \alpha)} \frac{1}{1 + W_1(x, \alpha)(E^2 + y^2) + W_2(x, \alpha)E^2} \end{aligned} \quad (3.50)$$

Integration over the variables E and y can be reduced to calculating the Poisson integral, but the calculation is different for each value of a and b for tensor t_{ab} . For example, we will show how to calculate such integrals for the component t_{00} , and we will keep in mind that the calculation of other components is similar. Therefore, the component t_{00} , where f_{ab} in

the integrand of 3.50 is equal to $f_{00} = (E^2 + x^2 s)$, has the form:

$$t_{00}^> = i \int_0^{\frac{1}{cP}} x dq \int_0^{2\pi} d\alpha \frac{1}{(w_1(x, \alpha))^2 w_2(x, \alpha) w_3(x, \alpha)} \times \\ \times \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dy (E^2 + x^2 s) \frac{1}{1 + W_1(x, \alpha)(E^2 + y^2) + W_2(x, \alpha)E^2} \quad (3.51)$$

Let us consider separately the calculation of integrals with respect to the variables E and y :

$$\int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dy (E^2 + x^2 s) \frac{1}{1 + W_1(x, \alpha)(E^2 + y^2) + W_2(x, \alpha)E^2} \quad (3.52)$$

Introducing the short notation for the sum $W_1(x, \alpha) + W_2(x, \alpha) = W_3(x, \alpha)$, the obtained result of integration is

$$\int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dy (E^2 + x^2 s) \frac{1}{1 + W_1(x, \alpha)(E^2 + y^2) + W_2(x, \alpha)E^2} = \\ = \frac{\pi}{\sqrt{W_3(x, \alpha)} \sqrt{W_1(x, \alpha)}} \left(\frac{1}{2} \frac{1}{|W_3(x, \alpha)|} + x^2 s \right) \quad (3.53)$$

Substitution of the calculated integrals 3.53 into expression 3.51 leads to

$$t_{00}^> = i\pi \int_0^{\frac{1}{cP}} x dq \int_0^{2\pi} d\alpha \frac{1}{(w_1(x, \alpha))^2 w_2(x, \alpha) w_3(x, \alpha)} \times \\ \times \frac{1}{\sqrt{W_3(x, \alpha)} \sqrt{W_1(x, \alpha)}} \left(\frac{1}{2} \frac{1}{|W_3(x, \alpha)|} + x^2 s \right) \quad (3.54)$$

Thus, we obtained a two-dimensional integral which can also be calculated numerically. Similarly, all other nonzero components of the tensor $t_{ab}^>$ were calculated, and calculating expression for tensor 3.2 as $t_{ab} = t_{ab}^> + t_{ab}^<$ we can return to the analytical expression 3.1.

3.4 Calculation of analytical expressions of R. Feynman's diagram with one loop for other contractions of field operators

In the previous sections, we calculated an analytical expression for the simplest single-loop diagram Fig. 3.3. We must consider all possible contractions of bispinor and glueball field operators. For the simplest case of elastic proton scattering, the diagram in Fig. 3.3 is added to the single-loop diagram Fig. 3.4.

The analytical expression corresponding to the diagram in Fig. 3.4 has the form:

$$\begin{aligned}
A = & \frac{g^4}{(2\pi)^6} (\bar{v}_{v_4}^+(P_3))_{s_3} \gamma_{s_3 s_1}^b (v_{v_1}^-(P_1))_{s_1} (\bar{v}_{v_3}^+(P_4))_{s_4} \gamma_{s_4 s_2}^a (v_{v_2}^-(P_2))_{s_2} \times \\
& \times \delta((P_3 + P_4) - (P_1 + P_2)) \int d^4k (k_a - 2P_{1a}) (k_b - P_{4b} - P_{2b}) \times \\
& \times \frac{1}{M_p^2 - (k - P_1)^2 - i\varepsilon} \frac{1}{M_p^2 - (k - P_4)^2 - i\varepsilon} \times \\
& \times \frac{1}{M_G^2 - (k - (P_1 - P_3))^2 - i\varepsilon} \frac{1}{M_G^2 - k^2 - i\varepsilon}
\end{aligned} \tag{3.55}$$

Here, g is the effective coupling constant, M_p and M_G are the proton and glueball masses, respectively, $(\bar{v}_{v_3}^+(P_4))_{s_4}, (v_{v_2}^-(P_2))_{s_2}, (\bar{v}_{v_4}^+(P_3))_{s_3},$

$(v_{v_1}^-(P_1))_{s_1}$ - solution of the Dirac equations, $\gamma_{s_4 s_2}^a, \gamma_{s_3 s_1}^b$ - elements of the Dirac matrices. All quantities are scaled to the proton mass. M_G and g are treated as fitted parameters.

As in the previous sections, we are interested in the calculation of the tensor, which we will call t_{ab}^1

$$t_{ab}^1 = \int d^4k (k_a - 2P_{1a}) (k_b - P_{4b} - P_{2b}) \frac{1}{M_p^2 - (k - P_1)^2 - i\varepsilon} \times \\ \times \frac{1}{M_p^2 - (k - P_4)^2 - i\varepsilon} \frac{1}{M_G^2 - (k - (P_1 - P_3))^2 - i\varepsilon} \frac{1}{M_G^2 - k^2 - i\varepsilon} \quad (3.56)$$

To calculate 3.56, it is convenient to use the coordinate system shown in Fig. 3.5. Describing each component of the tensor t_{ab}^1 , where the indices a and b run through 4 values from 0 to 3, we obtain 9 nonzero components, each of which will be calculated separately. Similarly to the tensor t_{ab} , all nine nonzero components of t_{ab}^1 have the same denominator but different numerators. We will show the solution of these components for the general case, since the main ideas of the calculation method are related to the form of the denominators. Let us denote the numerator $(k_a - 2P_{1a})(k_b - P_{4b} - P_{2b})$ as $f_1(k_0, \vec{k})$, then expression 3.56 takes the

form:

$$\begin{aligned}
t_{ab}^1 = & \int_{-\infty}^{\infty} dk_0 \int d\vec{k} f_1(k_0, \vec{k}) \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} - k_0\right)^2 + \left(\vec{P}_1 - \vec{k}\right)^2 - i\varepsilon} \times \\
& \times \frac{1}{M_p^2 - \left(\frac{\sqrt{s}}{2} - k_0\right)^2 + \left(\vec{P}_4 - \vec{k}\right)^2 - i\varepsilon} \frac{1}{M_G^2 - k_0^2 + \left(\left(\vec{P}_1 - \vec{P}_3\right) - \vec{k}\right)^2 - i\varepsilon} \\
& \times \frac{1}{M_G^2 - k_0^2 + \vec{k}^2 - i\varepsilon}
\end{aligned} \tag{3.57}$$

As you can see, the expression for tensor 3.57 is quite similar to the expression for tensor 3.2 and you will see that calculation for this tensor is indeed similar to the previous calculations. But we need to make sure about poles for obtained denominators and then make other calculations. For such a purpose, we will use the same substitution of variables 3.6 and write the obtained expression in such a way that we can see the location of the poles

$$\begin{aligned}
t_{ab}^1 = & \int_{-\infty}^{\infty} dq_0 \int d\vec{q} f_{ab}^1(q_0, \vec{q}) \frac{1}{q_0 - \left(\frac{\sqrt{s}}{2} + \sqrt{M_p^2 + \left(\frac{\vec{P}_1 + \vec{P}_3}{2} + \vec{q}\right)^2} - i\varepsilon\right)} \\
& \frac{1}{q_0 - \left(\frac{\sqrt{s}}{2} - \sqrt{M_p^2 + \left(\frac{\vec{P}_1 + \vec{P}_3}{2} + \vec{q}\right)^2} - i\varepsilon\right)}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{q_0 - \left(\frac{\sqrt{s}}{2} + \sqrt{M_p^2 + \left(\frac{\vec{P}_1 + \vec{P}_3}{2} - \vec{q} \right)^2 - i\varepsilon} \right)} \\
& \frac{1}{q_0 - \left(\frac{\sqrt{s}}{2} - \sqrt{M_p^2 + \left(\frac{\vec{P}_1 + \vec{P}_3}{2} - \vec{q} \right)^2 - i\varepsilon} \right)} \\
& \frac{1}{q_0 - \left(\sqrt{M_G^2 + \left(\frac{\vec{P}_1 - \vec{P}_3}{2} + \vec{q} \right)^2 - i\varepsilon} \right)} \frac{1}{q_0 - \left(-\sqrt{M_G^2 + \left(\frac{\vec{P}_1 - \vec{P}_3}{2} + \vec{q} \right)^2 - i\varepsilon} \right)} \\
& \frac{1}{q_0 - \left(\sqrt{M_G^2 + \left(\frac{\vec{P}_1 - \vec{P}_3}{2} - \vec{q} \right)^2 - i\varepsilon} \right)} \frac{1}{q_0 - \left(-\sqrt{M_G^2 + \left(\frac{\vec{P}_1 - \vec{P}_3}{2} - \vec{q} \right)^2 - i\varepsilon} \right)}
\end{aligned} \tag{3.58}$$

Values $\pm\eta = \pm\sqrt{M_G^2 + \left(\frac{\vec{P}_1 - \vec{P}_3}{2} \pm \vec{q} \right)^2 - i\varepsilon}$ are of particular interest. Consider them in the complex plane for the variable q_0 (Fig. 3.8).

As for the case showed in the previous section, the location of these quantities will depend on whether $\eta > \frac{\sqrt{s}}{2}$ or $\eta < \frac{\sqrt{s}}{2}$. If we consider the second option, when $\eta < \frac{\sqrt{s}}{2}$, then the poles will be located in the first and fourth quadrants. But if we choose the first option when $\eta > \frac{\sqrt{s}}{2}$, then the poles fall into the second and fourth quadrants, and we will be able to make the Wick rotation. From the inequality $\eta > \frac{\sqrt{s}}{2}$ we can see that $|\vec{q}| > 2P$, taking into account $|\vec{P}_1 + \vec{P}_3| = \sqrt{4P^2 - |t|}$. Thus, in order to avoid the problem with the previously described apparent singularity, in expression 3.57 it is necessary to divide the domain of integration over the components of the vector $\vec{q}$ into a finite part and a part over which

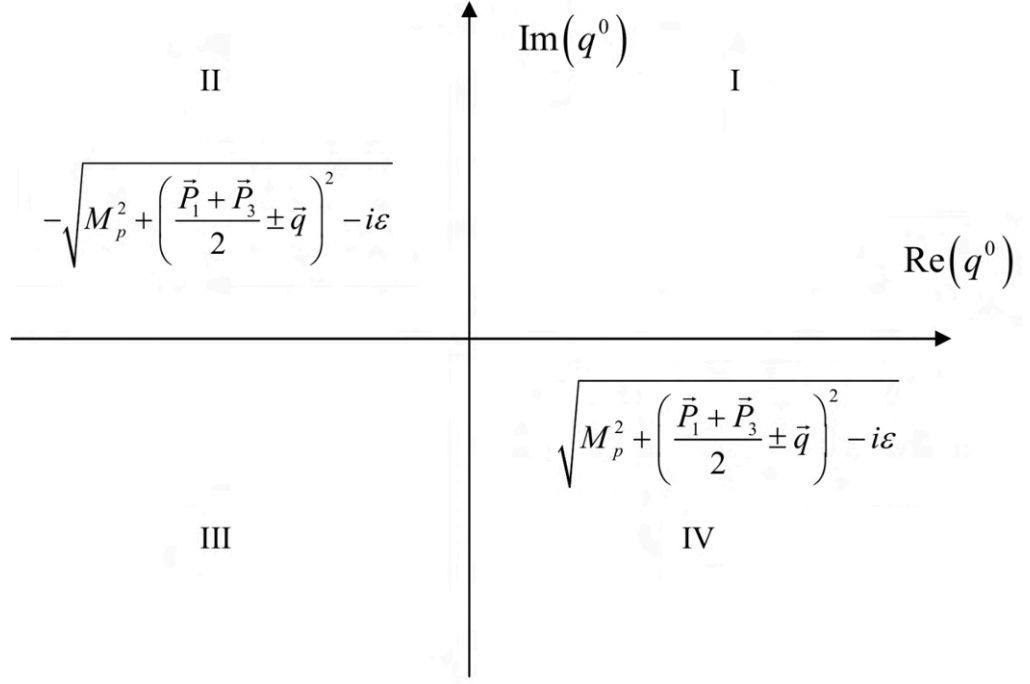


FIGURE 3.8: Consideration of quantities η and $-\eta$ on the complex plane

a Wick rotation is possible. Therefore, as in the previous sections, we divide the integral through which the tensor t_{ab}^1 is expressed through two integrals, using 3.9, we obtain an expression for the integral over a finite

domain

$$\begin{aligned}
t_{ab}^1 &= \int_{-\infty}^{\infty} dq_0 \int_{|\vec{q}| < 2P} d\vec{q} f_{ab}^1(q_0, \vec{q}) \\
&\frac{1}{\left(q_0 - \frac{\sqrt{s}}{2}\right)^2 - M_p^2 - (q^1)^2 - (q^2)^2 - \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3\right)^2 + i\varepsilon} \\
&\frac{1}{\left(q_0 - \frac{\sqrt{s}}{2}\right)^2 - M_p^2 - (q^1)^2 - (q^2)^2 - \left(\frac{\sqrt{4P^2 - |t|}}{2} - q^3\right)^2 + i\varepsilon} \quad (3.59) \\
&\frac{1}{(q_0)^2 - M_G^2 - \left(\frac{\sqrt{|t|}}{2} + q^1\right)^2 - (q^2)^2 - (q^3)^2 + i\varepsilon} \\
&\frac{1}{(q_0)^2 - M_G^2 - \left(\frac{\sqrt{|t|}}{2} - q^1\right)^2 - (q^2)^2 - (q^3)^2 + i\varepsilon}
\end{aligned}$$

and a domain where we can make a Wick rotation

$$\begin{aligned}
t_{ab}^1 &= \int_{-\infty}^{\infty} dq_0 \int_{|\vec{q}| > 2P} d\vec{q} f_{ab}^1(q_0, \vec{q}) \\
&\frac{1}{\left(q_0 - \frac{\sqrt{s}}{2}\right)^2 - M_p^2 - (q^1)^2 - (q^2)^2 - \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3\right)^2 + i\varepsilon} \\
&\frac{1}{\left(q_0 - \frac{\sqrt{s}}{2}\right)^2 - M_p^2 - (q^1)^2 - (q^2)^2 - \left(\frac{\sqrt{4P^2 - |t|}}{2} - q^3\right)^2 + i\varepsilon} \quad (3.60) \\
&\frac{1}{(q_0)^2 - M_G^2 - \left(\frac{\sqrt{|t|}}{2} + q^1\right)^2 - (q^2)^2 - (q^3)^2 + i\varepsilon} \\
&\frac{1}{(q_0)^2 - M_G^2 - \left(\frac{\sqrt{|t|}}{2} - q^1\right)^2 - (q^2)^2 - (q^3)^2 + i\varepsilon}
\end{aligned}$$

Let us consider the calculation of the expressions 3.59. As in the previous sections, we will calculate the tensor using the Laplace method.

Therefore, let us consider the system of equations for the first two denominators:

$$\begin{cases} M_p^2 - \left(\frac{\sqrt{s}}{2} - q_0\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3\right)^2 = 0 \\ M_p^2 - \left(\frac{\sqrt{s}}{2} - q_0\right)^2 + (q^1)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} - q^3\right)^2 = 0 \end{cases} \quad (3.61)$$

The solutions $q^3 = 0$ and $q^0 = \pm \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}} + \frac{\sqrt{s}}{2}$ satisfy this system of equations. Note that q_0 has two solutions: positive and negative. If we substitute them in expression 3.59, the denominators of these solutions will take different values. Therefore, we will need to consider two expressions with positive and negative solutions separately. Let us call the expression with positive solution $t_{ab}^{1<+}$ and the expression with negative solution $-t_{ab}^{1<-}$, respectively. We will consider the calculation of one of these tensors, for example $t_{ab}^{1<+}$, and the other is calculated in a similar way.

Let us substitute the solutions of the system of equations 3.61 into expression 3.59, where $q^0 = \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}} + \frac{\sqrt{s}}{2}$ and make a substitution of variables:

$$\begin{aligned} q^0 &= \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}} + \frac{\sqrt{s}}{2} + \varepsilon E, \\ q^3 &= \varepsilon x \end{aligned} \quad (3.62)$$

Let us extract the square of the modulus from the product of the first two denominators so that we can extract ε . Then 3.59 takes the form:

$$\begin{aligned}
t_{ab}^{1<+} &= \int_{-\infty}^{\infty} d\varepsilon E \int_{-\frac{2P}{\varepsilon}}^{\frac{2P}{\varepsilon}} d\varepsilon x \int_{(q^1)^2 + (q^2)^2 < 4P^2 - (\varepsilon x)^2} dq^2 dq^1 f_{ab}^{1+}(q^1, q^2, E, x) \times \\
&\times \frac{1}{\varepsilon \sqrt{\left(-2EC - \varepsilon E^2 + x\sqrt{4P^2 - |t|} + \varepsilon x^2\right)^2 + 1}} \times \\
&\times \exp \left(-i \arccos \left(\frac{-2EC - \varepsilon E^2 + x\sqrt{4P^2 - |t|} + \varepsilon x^2}{\sqrt{\left(-2EC - \varepsilon E^2 + x\sqrt{4P^2 - |t|} + \varepsilon x^2\right)^2 + 1}} \right) \right) \times \\
&\times \frac{1}{\varepsilon \sqrt{\left(-2EC - \varepsilon E^2 - x\sqrt{4P^2 - |t|} + \varepsilon x^2\right)^2 + 1}} \times \\
&\times \exp \left(-i \arccos \left(\frac{-2EC - \varepsilon E^2 - x\sqrt{4P^2 - |t|} + \varepsilon x^2}{\sqrt{\left(-2EC - \varepsilon E^2 - x\sqrt{4P^2 - |t|} + \varepsilon x^2\right)^2 + 1}} \right) \right) \times \\
&\times \frac{1}{M_G^2 - \left(C + \frac{\sqrt{s}}{2} + \varepsilon E\right)^2 + \left(\frac{\sqrt{|t|}}{2} + q^1\right)^2 + (q^2)^2 + (\varepsilon x)^2 - i\varepsilon} \times \\
&\times \frac{1}{M_G^2 - \left(C + \frac{\sqrt{s}}{2} + \varepsilon E\right)^2 + \left(\frac{\sqrt{|t|}}{2} - q^1\right)^2 + (q^2)^2 + (\varepsilon x)^2 - i\varepsilon}
\end{aligned} \tag{3.63}$$

where the notation $C = \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}}$ is introduced to make the expression more compact. Now ε can be reduced in the numerator and denominator. Let us represent the first two denominators in the form of an exponent with the logarithm of the denominators. Then we

expand these exponents in a Taylor series to the second order:

$$\begin{aligned}
& \exp \left(\ln \left(\frac{1}{\sqrt{\left(-2EC + x\sqrt{4P^2 - |t|} + \varepsilon x^2\right)^2 + 1}} \times \right. \right. \\
& \quad \left. \left. \times \frac{1}{\sqrt{\left(-2EC - x\sqrt{4P^2 - |t|} + \varepsilon x^2\right)^2 + 1}} \right) \right) = \\
& = \exp \left(-4E^2C^2 - x^2(4P^2 - |t|) \right)
\end{aligned} \tag{3.64}$$

Similar for the other two factors:

$$\begin{aligned}
& \exp \left(-i \arccos \left(\frac{-2EC + x\sqrt{4P^2 - |t|}}{\sqrt{\left(-2EC + x\sqrt{4P^2 - |t|}\right)^2 + 1}} \right) \right) \times \\
& \times \exp \left(-i \arccos \left(\frac{-2EC - x\sqrt{4P^2 - |t|}}{\sqrt{\left(-2EC - x\sqrt{4P^2 - |t|}\right)^2 + 1}} \right) \right) = \\
& = \exp \left(i \left(\frac{\pi}{2} + 2EC - x\sqrt{4P^2 - |t|} \right) \right) \times \\
& \times \exp \left(i \left(\frac{\pi}{2} + 2EC + x\sqrt{4P^2 - |t|} \right) \right)
\end{aligned} \tag{3.65}$$

Now we return to the expression without using the notation $C = \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}}$ and calculate the integrals with respect to E and x .

Substituting the obtained expressions 3.64 and 3.65 into 3.63 we can separate the integration over the variables E and x . The integral over

the variable x is the Poisson integral: $\int_{-\infty}^{\infty} dx \exp(-x^2(4P^2 - |t|)) = \frac{\sqrt{\pi}}{\sqrt{4P^2 - |t|}}$, and the integral over the variable E can be reduced to the Poisson integral:

$$\begin{aligned} & \int_{-\infty}^{\infty} dE \exp\left(-4E^2 \left(M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}\right) + \right. \\ & \left. + 4iE \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}}\right) = \\ & \frac{\sqrt{\pi}}{2e \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}}} \end{aligned} \quad (3.66)$$

Taking into account the obtained expressions, formula 3.63 takes the form

$$\begin{aligned} t_{ab}^{1<+} = & -\frac{\pi}{2e \sqrt{4P^2 - |t|}} \int_{(q^1)^2 + (q^2)^2 < 4P^2}^{\infty} dq^2 dq^1 \frac{f_{ab}^{1+}(q^1, q^2)}{2 \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}}} \\ & \frac{1}{\left(M_G^2 - M_p^2 - \frac{s}{4} - P^2 - \sqrt{s} \sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}} + \frac{|t|}{4}\right)^2 - (q^1)^2 |t|} \end{aligned} \quad (3.67)$$

After passing to the polar coordinates in expression 3.67

$$\begin{aligned} q^1 &= q \cos \alpha, \\ q^2 &= q \sin \alpha \end{aligned} \quad (3.68)$$

the obtained expression can be integrated over the variable α and after such an integration, one-dimensional integral over the variable q we can

calculate numerically:

$$\begin{aligned}
t_{ab}^{1<+} = & -\frac{\pi}{2e\sqrt{4P^2 - |t|}} \int_0^{2P} q dq \frac{f_{ab}^{1+}(q)}{\sqrt{M_p^2 + q^2 + \frac{4P^2 - |t|}{4}}} \\
& \frac{1}{\left(-M_G^2 + M_p^2 + \frac{s}{4} + P^2 + \sqrt{s} \sqrt{M_p^2 + q^2 + \frac{4P^2 - |t|}{4}} - \frac{|t|}{4} \right)} \\
& \frac{1}{\sqrt{\left(M_G^2 - M_p^2 - \frac{s}{4} - P^2 - \sqrt{s} \sqrt{M_p^2 + q^2 + \frac{4P^2 - |t|}{4}} + \frac{|t|}{4} \right)^2 - q^2 |t|}}
\end{aligned} \tag{3.69}$$

Similarly, all components $t_{ab}^{1<-}$ were calculated with the substitution of the negative solution $q_0 = -\sqrt{M_p^2 + (q^1)^2 + (q^2)^2 + \frac{4P^2 - |t|}{4}} + \frac{\sqrt{s}}{2}$.

We will consider the calculation of expression 3.60 after the Wick rotation later. Firstly, we are going to consider the calculation of expression 3.59 for the other two denominators. Just as in the previous cases, let us write a system of equations for the real parts of these denominators:

$$\begin{cases} M_G^2 + (q_0)^2 + \left(\frac{\sqrt{|t|}}{2} + q^1 \right)^2 + (q^2)^2 + (q^3)^2 = 0, \\ M_G^2 + (q_0)^2 + \left(\frac{\sqrt{|t|}}{2} - q^1 \right)^2 + (q^2)^2 + (q^3)^2 = 0 \end{cases} \tag{3.70}$$

Solutions satisfying this system of equations are $q_1 = 0$ and $q_0 = \pm \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2}$. Let us represent in expression 3.59 the product of fractions containing the corresponding denominators using

the modulus and argument and then introduce new variables

$$\begin{aligned} q_0 &= \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2 + \varepsilon E}, \\ q^1 &= \varepsilon x \end{aligned} \tag{3.71}$$

We will call the resulting expression $t_{ab}^{2<+}$, which tells us that we calculate the contribution to the tensor components from a finite part of the integration domain with the substitution of a positive solution for q_0 . This tensor has the form

$$\begin{aligned} t_{ab}^{2<+} &= \int_{-\infty}^{\infty} dE \int_{-\frac{2P}{\varepsilon}}^{\frac{2P}{\varepsilon}} dx \int_{(q^2)^2 + (q^3)^2 < 4P^2 - (\varepsilon x)^2}^{\infty} dq^2 dq^3 f_{ab}^{2+}(E, x, q^2, q^3) \times \\ &\quad \frac{1}{M_p^2 - B^2 + (\varepsilon x)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} + q^3\right)^2 - i\varepsilon} \\ &\quad \frac{1}{M_p^2 - B^2 + (\varepsilon x)^2 + (q^2)^2 + \left(\frac{\sqrt{4P^2 - |t|}}{2} - q^3\right)^2 - i\varepsilon} \\ &\quad \frac{1}{\sqrt{\left(-2E\sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2} - \varepsilon E^2 - \sqrt{|t|x + \varepsilon x^2}\right)^2 + 1}} \end{aligned}$$

$$\begin{aligned}
& \frac{1}{\exp \left(-i \arccos \left(\frac{-2E \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2 - \varepsilon E^2 - \sqrt{|t|x + \varepsilon x^2}}}{\sqrt{\left(-2E \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2 - \varepsilon E^2 - \sqrt{|t|x + \varepsilon x^2}} \right)^2 + 1}} \right) \right)} \\
& \frac{1}{\sqrt{\left(-2E \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2 - \varepsilon E^2 + \sqrt{|t|x + \varepsilon x^2}} \right)^2 + 1}} \\
& \frac{1}{\exp \left(-i \arccos \left(\frac{-2E \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2 - \varepsilon E^2 + \sqrt{|t|x + \varepsilon x^2}}}{\sqrt{\left(-2E \sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2 - \varepsilon E^2 + \sqrt{|t|x + \varepsilon x^2}} \right)^2 + 1}} \right) \right)}
\end{aligned} \tag{3.72}$$

where the notation $B = \left(\frac{\sqrt{s}}{2} - \left(\sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2} + \varepsilon E \right) \right)$ is introduced for a more compact notation.

Here $f_{ab}^{2<+}(E, x, q^2, q^3)$ are generally denoted as numerators for various components of the tensor $t_{ab}^{2<+}$. We introduced a change of variables 3.71 in order to take ε out of brackets in the numerator and denominator of 3.72 and reduce it, as was done before. Then in 3.72 we can direct ε to zero. After that, we represent the third and fourth denominators in the form of exponents in the logarithm of these denominators and make a Taylor expansion of them to the second order.

We substitute the obtained expressions into 3.72 and we obtain that integration with the respect to the variables E and x reduces to Poisson integrals. After all these transformations, the following form of expression

3.72 becomes

$$t_{ab}^{2<+} = - \frac{\pi}{e\sqrt{|t|}} \int_{(q^2)^2 + (q^3)^2 < 4P^2}^{\infty} dq^2 dq^3 \frac{f_{ab}^{2+}(q^2, q^3)}{2\sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2}} \frac{1}{\left(\sqrt{s}\sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2} - M_G^2 - \frac{|t|}{2}\right)^2 - (q^3)^2(4P^2 - |t|)} \quad (3.73)$$

As before, it is convenient to switch to polar coordinates

$$\begin{aligned} q^2 &= q \cos \alpha, \\ q^3 &= q \sin \alpha \end{aligned} \quad (3.74)$$

After switching to polar coordinates, expression 3.73 can be integrated analytically with respect to the variable α , then we obtain a one-dimensional integral with respect to the variable q , which was calculated numerically

$$t_{ab}^{2<+} = - \frac{\pi}{e\sqrt{|t|}} \int_0^{2P} q dq \frac{f_{ab}^{2+}(q)}{\sqrt{M_G^2 + \frac{|t|}{4} + q^2}} \frac{1}{\left(\sqrt{s}\sqrt{M_G^2 + \frac{|t|}{4} + q^2} - M_G^2 - \frac{|t|}{2}\right)} \frac{1}{\sqrt{\left(\sqrt{s}\sqrt{M_G^2 + \frac{|t|}{4} + q^2} - M_G^2 - \frac{|t|}{2}\right)^2 - q^2(4P^2 - |t|)}} \quad (3.75)$$

Similarly, the components of the tensor $t_{ab}^{2<-}$ with a negative solution $q_0 = -\sqrt{M_G^2 + \frac{|t|}{4} + (q^2)^2 + (q^3)^2}$ were calculated.

We have calculated the contributions of the integral over a finite domain to the components of the tensor using the Laplace method for the

diagram in Fig. 3.4. Now it remains to consider the calculation of the tensor 3.60 after the Wick rotation, which will lead to the replacement of q_0 in the integrand by iq_0 . In the resulting expression, we can immediately approximate ε to zero because this will no longer lead to a singularity of the integrand inside the integration domain. The components of the tensor $t_{ab}^{1>}$ were calculated separately as in the previous cases. The general form of the numerators of these components will be denoted by $f_{ab}^{1>}$. Expression 3.60 takes the form

$$\begin{aligned}
t_{ab}^{1>} = & i \int_{-\infty}^{\infty} dq_0 \int_{-\infty}^{\infty} dq^2 \int_{(q^1)^2 + (q^3)^2 > 4P^2}^{\infty} \frac{dq^1 dq^3 f_{ab}^{1>}(q_0, q^1, q^2, q^3)}{1} \\
& \frac{1}{(q_0)^2 + iq_0\sqrt{s} - \frac{s}{4} + M_p^2 + (q^1)^2 + (q^2)^2 + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3\right)^2} \\
& \frac{1}{(q_0)^2 + iq_0\sqrt{s} - \frac{s}{4} + M_p^2 + (q^1)^2 + (q^2)^2 + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3\right)^2} \\
& \frac{1}{(q_0)^2 + M_G^2 + \left(\frac{\sqrt{|t|}}{2} + q^1\right)^2 + (q^2)^2 + (q^3)^2} \\
& \frac{1}{(q_0)^2 + M_G^2 + \left(\frac{\sqrt{|t|}}{2} - q^1\right)^2 + (q^2)^2 + (q^3)^2}
\end{aligned} \tag{3.76}$$

Since at $q_0 = 0$ and $q^2 = 0$ the maximum modulus of the integrand is reached, we have divided the integration over the variables q_0 and q^2 , and over q^1 and q^3 . Let us consider the calculation of expression 3.76 for the component $t_{00}^{1>}$

$$\begin{aligned}
t_{00}^{1>} &= i s \int_{(q^1)^2 + (q^3)^2 > 4P^2}^{\infty} dq^1 dq^3 \frac{1}{\left((q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3 \right)^2 \right)} \\
&\frac{1}{\left((q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3 \right)^2 \right)} \frac{1}{\left(M_G^2 + \left(\frac{\sqrt{|t|}}{2} + q^1 \right)^2 + (q^3)^2 \right)} \\
&\frac{1}{\left(M_G^2 + \left(\frac{\sqrt{|t|}}{2} - q^1 \right)^2 + (q^3)^2 \right)} \times \\
&\times \int_{-\infty}^{\infty} dq_0 \int_{-\infty}^{\infty} dq^2 \frac{1}{\left(1 + \frac{(q_0)^2 + (q^2)^2 + i q_0 \sqrt{s}}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3 \right)^2} \right)} \times \\
&\times \frac{1}{\left(1 + \frac{(q_0)^2 + (q^2)^2}{M_G^2 + \left(\frac{\sqrt{|t|}}{2} + q^1 \right)^2 + (q^3)^2} \right)} \times \\
&\times \frac{1}{\left(1 + \frac{(q_0)^2 + (q^2)^2 + i q_0 \sqrt{s}}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3 \right)^2} \right)} \frac{1}{\left(1 + \frac{(q_0)^2 + (q^2)^2}{M_G^2 + \left(\frac{\sqrt{|t|}}{2} - q^1 \right)^2 + (q^3)^2} \right)} \\
&\hspace{15em} (3.77)
\end{aligned}$$

Let us consider separately the integration with respect to the variables q_0 and q^2 . As before, we represent the denominators as exponents in a logarithm and expand the exponent in a Taylor series to the second order. Then we group separately the factors at $(q_0)^2$ and at $i q_0$ and $(q^2)^2$. Let us introduce some notations to shorten the expression for the factor at $(q_0)^2$

$$\begin{aligned}
w_1(q^1, q^3) = & \frac{1}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3\right)^2} + \\
& + \frac{s}{2 \left((q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3\right)^2 \right)} + \\
& + \frac{1}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3\right)^2} + \frac{1}{M_G^2 + \left(\frac{\sqrt{|t|}}{2} + q^1\right)^2 + (q^3)^2} \\
& + \frac{s}{2 \left((q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3\right)^2 \right)} + \\
& + \frac{1}{M_G^2 + \left(\frac{\sqrt{|t|}}{2} - q^1\right)^2 + (q^3)^2}
\end{aligned} \tag{3.78}$$

for the factor at iq_0

$$\begin{aligned}
w_2(q^1, q^3) = & \frac{1}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3\right)^2} + \\
& + \frac{1}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3\right)^2} \\
& + \frac{1}{M_G^2 + \left(\frac{\sqrt{|t|}}{2} + q^1\right)^2 + (q^3)^2} + \frac{1}{M_G^2 + \left(\frac{\sqrt{|t|}}{2} - q^1\right)^2 + (q^3)^2}
\end{aligned} \tag{3.79}$$

and for the factor at $(q^2)^2$

$$\begin{aligned}
w_3(q^1, q^3) = & \frac{\sqrt{s}}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3 \right)^2} + \\
& + \frac{\sqrt{s}}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3 \right)^2}
\end{aligned} \tag{3.80}$$

Taking into account 3.78, 3.79 and 3.80, the expression containing integration with respect to the variables q_0 and q^2 reduces to the Poisson integral and takes the form

$$\begin{aligned}
& \int_{-\infty}^{\infty} dq_0 \int_{-\infty}^{\infty} dq^2 \frac{1}{\left(1 + \frac{(q_0)^2 + (q^2)^2 + iq_0 \sqrt{s}}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3 \right)^2} \right)} \frac{1}{\left(1 + \frac{(q_0)^2 + (q^2)^2}{M_G^2 + \left(\frac{\sqrt{|t|}}{2} + q^1 \right)^2 + (q^3)^2} \right)} \\
& \frac{1}{\left(1 + \frac{(q_0)^2 + (q^2)^2 + iq_0 \sqrt{s}}{(q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3 \right)^2} \right)} \frac{1}{\left(1 + \frac{(q_0)^2 + (q^2)^2}{M_G^2 + \left(\frac{\sqrt{|t|}}{2} - q^1 \right)^2 + (q^3)^2} \right)} = \\
& = \int_{-\infty}^{\infty} dq_0 \exp \left(-w_1(q^1, q^3) (q_0)^2 - w_3(q^1, q^3) iq_0 \right) \int_{-\infty}^{\infty} dq^2 \times \\
& \times \exp \left(-w_2(q^1, q^3) (q^2)^2 \right) = \frac{\pi}{\sqrt{w_2(q^1, q^3)} \sqrt{w_1(q^1, q^3)}} \times \\
& \times \exp \left(-\frac{1}{4} \frac{(w_3(q^1, q^3))^2}{w_1(q^1, q^3)} \right)
\end{aligned} \tag{3.81}$$

Let us substitute the obtained expression 3.81 into 3.77

$$\begin{aligned}
t_{00}^{1>} &= i s \pi \int_{(q^1)^2 + (q^3)^2 > 4P^2}^{\infty} dq^1 dq^3 \frac{1}{\left((q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} + q^3 \right)^2 \right)} \times \\
&\times \frac{1}{\left((q^1)^2 + M_p^2 - \frac{s}{4} + \left(\sqrt{P^2 - \frac{|t|}{4}} - q^3 \right)^2 \right)} \times \\
&\times \frac{1}{\left(M_G^2 + \left(\frac{\sqrt{|t|}}{2} + q^1 \right)^2 + (q^3)^2 \right)} \frac{1}{\left(M_G^2 + \left(\frac{\sqrt{|t|}}{2} - q^1 \right)^2 + (q^3)^2 \right)} \times \\
&\frac{1}{\sqrt{w_2(q^1, q^3)} \sqrt{w_1(q^1, q^3)}} \exp \left(-\frac{1}{4} \frac{(w_3(q^1, q^3))^2}{w_1(q^1, q^3)} \right)
\end{aligned} \tag{3.82}$$

Therefore, the component 3.82 remains to be integrated with respect to the variables q^1 and q^3 , which we calculated numerically. All other non-zero components of the tensor $t_{ab}^{1>}$ were calculated in the same way. All integrals that we calculated numerically were calculated using Fortran programming language.

3.5 Calculation results and discussion

In this section the differential cross section for the elastic proton scattering $\frac{d\sigma_{el}}{dt}(t)$ on the square of the transmitted four-momentum t was calculated. The calculations were done as follows:

- Calculation of the tensors that are included in the analytical expressions 3.1 and 3.55.

- Find the sum of these tensors and calculated the square of the scattering amplitude modulus and convolved with the density matrix.
- Add to the diagrams in Fig.3.3 and Fig.3.4 diagrams with the exchange of four-momentum P_3 and P_4 .
- Take into account the loopless diagrams of the lower order (tree), which are depicted in Fig. 3.2.

The model used has two dependent parameters M_G - the mass of the glueball and G - the strong interaction constant. In the calculations, all quantities were dimensionless to the proton mass M_P , which is equal to 0.938 GeV. The dependence of $\frac{d\sigma_{el}}{dt}(t)$ on t is shown in the plots (Fig. 3.9 - 3.12b) at different energies $\sqrt{s}$. Comparison with the experiment has shown agreement so far only at a qualitative level, but we expect that taking into account diagrams with a larger number of loops will lead to a quantitative agreement with the experimental data [70, 73, 74].

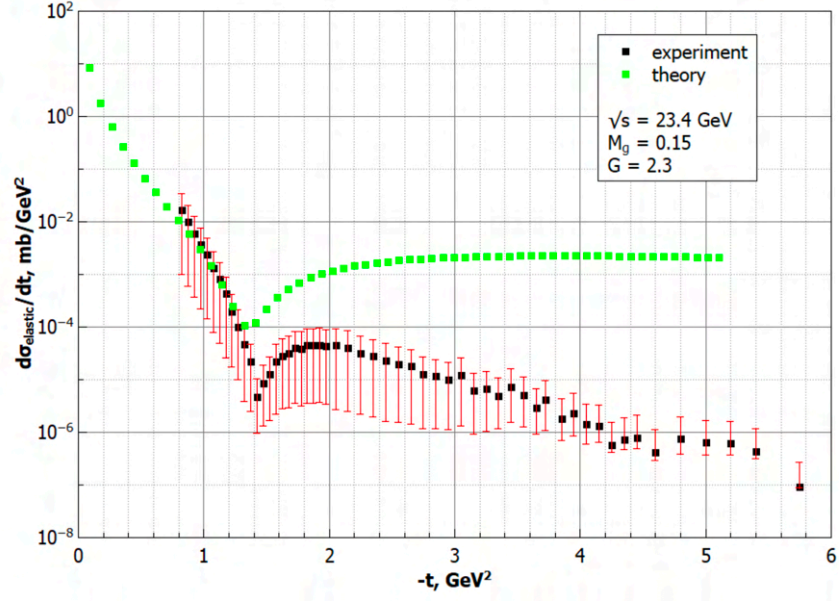


FIGURE 3.9: Comparison of the dependence of the differential cross section $\frac{d\sigma_{el}}{dt}(t)$ on the square of the transmitted four-momentum t at the energy $\sqrt{s} = 23.4$ GeV, calculated using the model described in this work (green dots) with the experiment (black dots) [19]

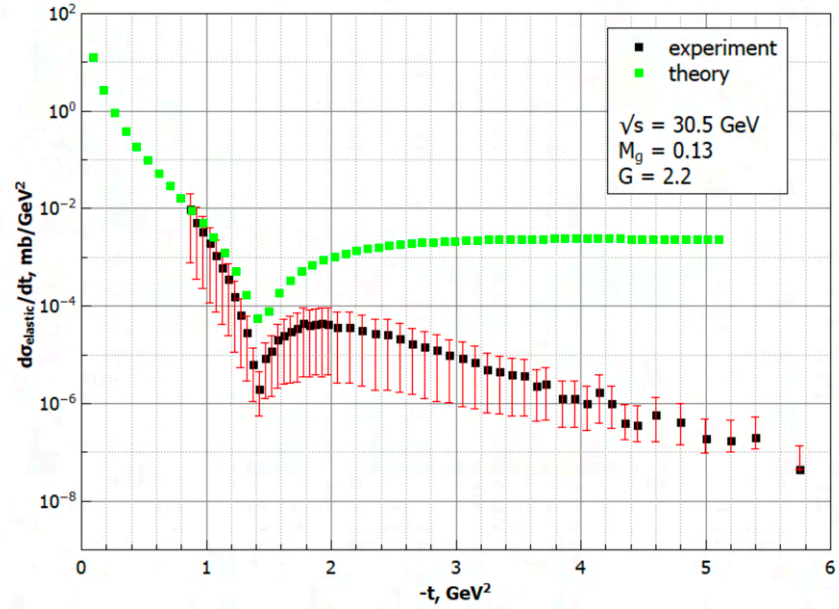


FIGURE 3.10: Comparison of the dependence of the differential cross section $\frac{d\sigma_{el}}{dt}(t)$ on the square of the transmitted four-momentum t at the energy $\sqrt{s} = 30.5$ GeV, calculated using the model described in this work (green dots) with the experiment (black dots) [19]

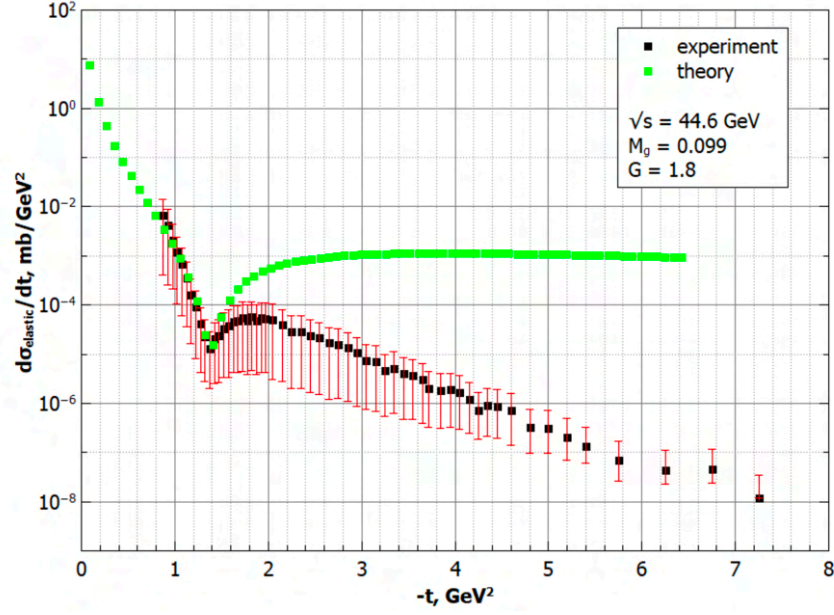


FIGURE 3.11: Comparison of the dependence of the differential cross section $\frac{d\sigma_{el}}{dt}(t)$ on the square of the transmitted four-momentum t at the energy $\sqrt{s} = 44.6$ GeV, calculated using the model described in this work (green dots) with the experiment (black dots) [19]

As you can see, the plots shown above are made with different values of the flexible parameters M_G and G (Fig.3.9, 3.10, 3.11). We tried to made plots for different values of energies, but with the same mass of the glueball $M_G = 0.15$ and the strong interaction constant $G = 2.3$ (Fig. 3.12). Based on the plots we can conclude, that mass of the glueball M_G and the strong interaction constant G are different for different energies.

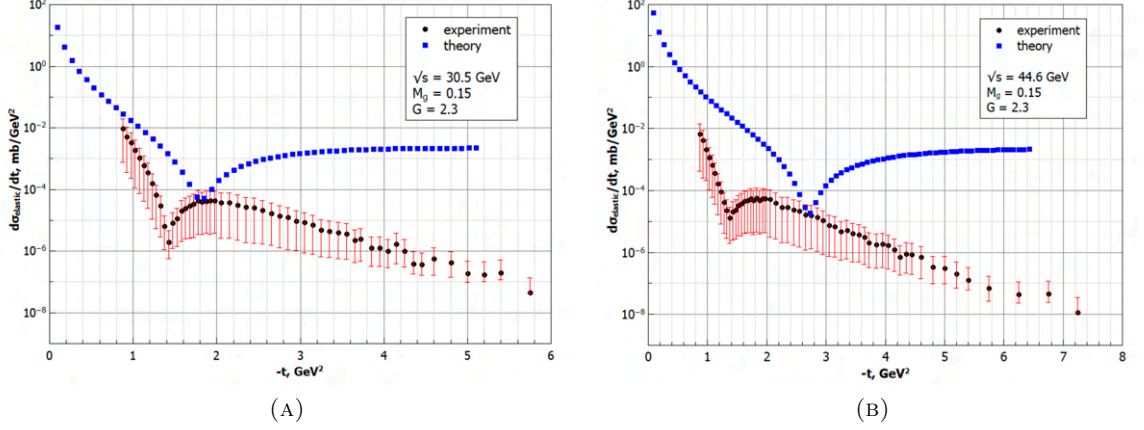


FIGURE 3.12: Comparison of the dependence of the differential cross section $\frac{d\sigma_{el}}{dt}(t)$ on the square of the transmitted four-momentum t at the energies $\sqrt{s} = 30.5$ GeV (A) and $\sqrt{s} = 44.6$ GeV, calculated using the model described in this work (green dots) with the experiment (black dots) [19] when parameters M_G and G are same

In order to show the qualitative behavior of the dependence of the differential cross section on the square of the transmitted four-momentum we have separately plotted the dependence of the differential cross section on the square of the transmitted four-momentum, where t is plotted up to 35 GeV^2 (Fig. 3.13).

At the same time with the described calculations, we performed similar calculations in the ϕ^3 model with scalar fields, in which we did not obtain non-monotonicity effects similar to those shown in the above figures. This allows us to conclude that the physical mechanisms that lead to non-monotonicity are associated precisely with spin effects. The spin-flip effects in proton scattering within the framework of the region theory were calculated in [76]. However, our results differ from presented in cited work. The difference lies in the fact that within the Regge approach, non-monotonicity effects arise due to the exchange of reggeons with different signatures, i.e., due to reggeon contributions to the scattering amplitude with different signs. Spin effects in [76] lead only to

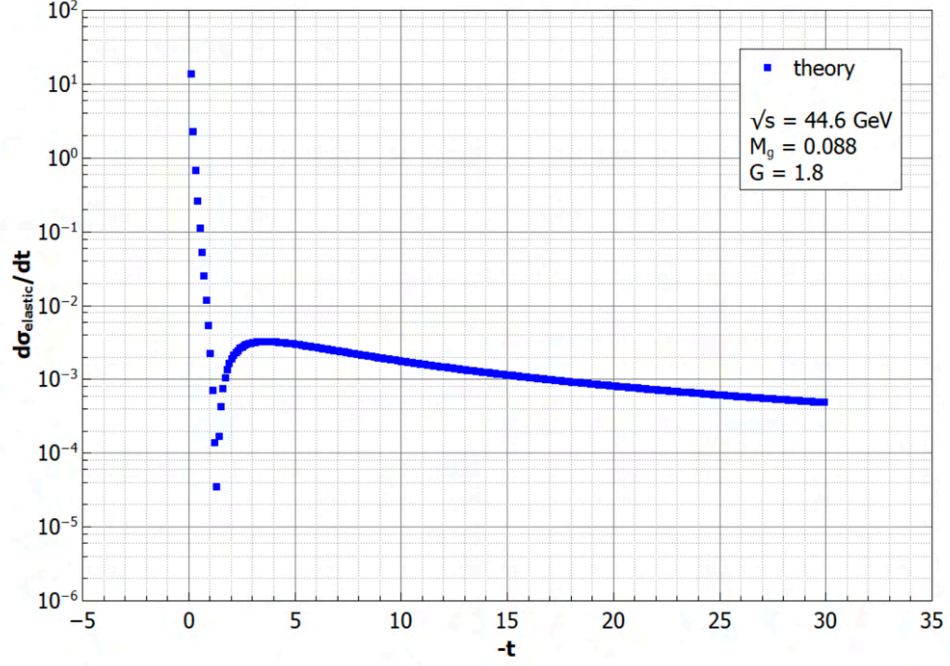


FIGURE 3.13: The dependence of the differential cross section $\frac{d\sigma_{el}}{dt}(t)$ on the square of the transmitted four-momentum t at the energies $\sqrt{s} = 44.6$ GeV. The plot on the axis of the transmitted four-momentum t is extended to show the behavior of the result

quantitative changes in the dependence of the cross section on the square of the transmitted four-momentum. In the model we use the spin effects lead to non-monotonicity effects.

3.6 Summary

The work shows that the method of multiparticle fields leads to dynamical models that can be used to describe experiments on scattering of multi-quark systems and achieve correspondence with experimental data at least at the level of qualitative agreement.

The application of the Laplace method to calculate loop diagrams allows their approximate calculation, but is still quite cumbersome and requires further improvement.

The experimentally observed effects of non-monotonicity of the dependence of the differential cross section of elastic proton scattering on the square of the transferred four-momentum in our model are consequences of spin effects. Taking into account these effects in loopless and simplest loop diagrams led to a qualitative agreement with experiment in the dependence of the cross section on the square of the transmitted four-momentum. The obtained results suggest that further consideration of more complex loop diagrams with more than one loop will allow achieving quantitative agreement with the experimental results.

Chapter 4

Tools for understanding SIDIS measurements

Although the Parton Model successfully described the scaling behavior of structure functions in deep inelastic scattering (DIS) - suggesting that high-energy electrons interacted with point-like constituents inside protons - its limitations became evident in semi-inclusive DIS (SIDIS). The naive Parton Model neglects gluon radiation, parton splitting, and soft interactions, and it assumes quarks carry only longitudinal momentum ($\mathbf{k}_T = 0$). These omissions are critical because the observed transverse momentum spectrum ($\mathbf{P}_{hT}$) of the detected hadron cannot be explained without accounting for intrinsic quark transverse momentum ($\mathbf{k}_T$) and soft gluon emissions. Furthermore, asymmetries such as the Collins and Sivers effects arise from couplings between parton $\mathbf{k}_T$ and hadron $\mathbf{p}_T$,

which are absent in collinear parton distribution functions (PDFs). Additionally, while the Parton Model predicts no Q^2 -dependence, experimental data exhibit clear logarithmic scaling violations. Given these shortcomings, it became necessary to describe SIDIS within the framework of QCD, employing factorization theorems to systematically separate perturbative and non-perturbative contributions. These issues are systematically addressed in QCD through the Collins-Soper-Sterman (CSS) factorization framework [77, 78] and transverse-momentum-dependent (TMD) formalism [79–84], which incorporate perturbative resummation and non-perturbative TMD distributions.

4.1 SIDIS kinematics

Semi-Inclusive Deep Inelastic Scattering provides a powerful tool for probing the internal structure of hadrons through the study of hadron production in the final state. The differential cross section for this process is given by:

$$\frac{d\sigma}{dQ^2 dx_{Bj} dz_h d\mathbf{P}_{hT}} \quad (4.1)$$

where x_{Bj} , z_h and Q are the usual SIDIS kinematic variables (introduced in eq.2.1) and $\mathbf{P}_{hT}$ is the transverse momentum of the detected hadron in the Breit frame (appendix C). The key challenge lies in identifying the kinematic regimes where transverse-momentum-dependent (TMD) factorization remains valid, particularly when hadron production is described via fragmentation functions. The TMD framework was experimentally validated through measurements of Collins/Sivers asymmetries

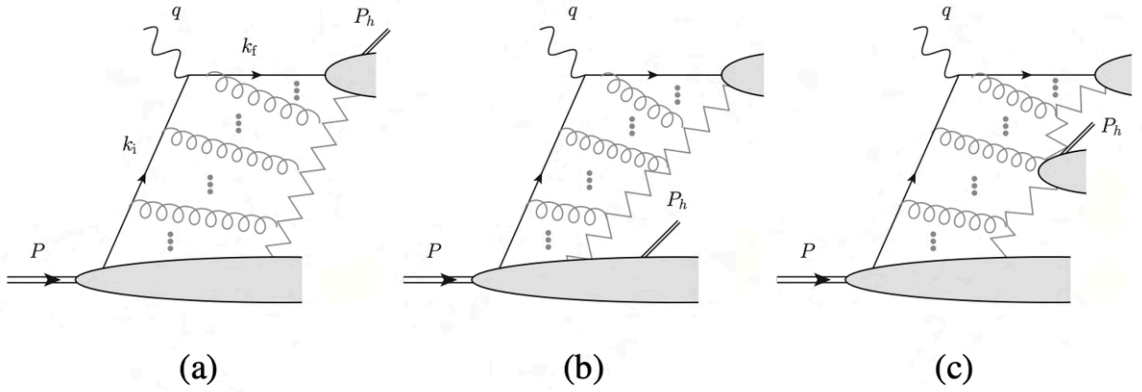


FIGURE 4.1: Lowest order SIDIS diagrams that correspond to (a) the current region (b) the target region and (c) the central (soft) region. Faded zigzags indicate non-perturbative interactions and hadronization between the final-state parton and target jet

in SIDIS by HERMES [85–87] and COMPASS [88–90], and in e^+e^- annihilation by BELLE [91, 92]. Later, STAR and PHENIX confirmed TMD universality in proton-proton collisions [93, 94].

For the photon virtuality Q^2 to serve as a hard scale, it must satisfy $Q^2 \gg \Lambda_{QCD}^2$. Meanwhile, as it was shown in [95], the hadron’s transverse momentum $\mathbf{P}_{hT}$ plays a crucial role in distinguishing different kinematic regimes. When $\mathbf{P}_{hT}$ is small compared to Q , the process can be categorized into three primary regions: current fragmentation, target fragmentation, and the central (soft) region. These regions are illustrated in Fig. 4.1, where the virtual photon interacts with a quark/parton, leading to large-angle scattering and subsequent hadronization.

The zigzag lines and additional gluons depicted in Fig. 4.1 serve as an important reminder that the simplest Parton Model diagrams provide only a schematic representation of the full QCD dynamics, especially regarding final state interactions. In reality, the extra gluon exchanges - absent in naive parton model graphs - manifest as Wilson line attachments

in the operator definitions of parton densities, fragmentation functions, and other non-perturbative matrix elements upon making the necessary approximations for factorization. A more complete description might also include mechanisms like string-like fragmentation [96, 97], which clarify the boundaries between different kinematic regions [98].

The three regions illustrated in Fig. 4.1 are distinguished by the kinematics of the produced hadron, with each region governed by its own factorization theorem. Factorization is most accurate when its simplifications properly represent the physics of that specific regime. Crucially, all cases assume the high-energy limit where $Q^2 \gg \Lambda_{QCD}^2$ at fixed x_{Bj} and small $\mathbf{P}_{hT}$.

In the **Current Fragmentation Region** (Fig. 4.1a), the outgoing quark fragments into the observed hadron, denoted by the momentum P_h , which retains a momentum direction similar to the original quark. This regime is well-described by TMD factorization, incorporating both TMD PDFs and TMD FFs. Theoretical efforts have concentrated most heavily on this region, resulting in a mature framework of factorization theorems [4, 99–103]. It applies when, in addition to the conditions mentioned above, z_h large enough.

The **Target Fragmentation Region** (Fig. 4.1b) is characterized by the hadron being collinear to the target remnants. This regime is best described using extended fracture functions [104–109], particularly those that include transverse momentum dependence [109]. While the

formalism is well-established, the kinematic boundaries of its applicability remain an open question.

The **Central (Soft) Fragmentation Region** (Fig. 4.1c) involves hadron production at the rapidities, distinct from both the current and target regions (much less than for target region, much greater than for current region). Although a dedicated factorization theorem for this regime is not yet fully developed, it is expected to involve universal soft factors analogous to those in TMD factorization.

An important observation is that the current and target fragmentation regions each overlap with the central region. For instance, when the hadron rapidity y_h is moderately negative, both current and central region factorizations may apply simultaneously. A unified description requires matching these factorization schemes, similar to the treatment of transverse momentum in Drell-Yan processes [77]. For a complete description of SIDIS across all $\mathbf{P}_{hT}$, it is necessary to match collinear factorization with the unified TMD factorizations corresponding to the three low- $\mathbf{P}_{hT}$ regions.

4.1.1 SIDIS diagram with the partonic subprocess

To understand these kinematic regions at their most fundamental level, we focus our theoretical microscope on the partonic subprocess $k_i + q \rightarrow k_f + k_X$. Let us first consider the simplest case: a hadron produced collinearly with an outgoing parton - what we call a "current" hadron

(Fig. 4.2). Different implementations of the partonic subprocess, each relevant to particular kinematic domains, are presented in Fig. 4.3.

The “Berger criterion” [110] is often invoked in SIDIS to delineate the current fragmentation region. Unlike the z_h -only conditions proposed in [111], as it was mentioned while defining kinematical regions, the analysis in [95] highlights the necessity of including P_{hT} dependence. When extracting TMD fragmentation functions, it is crucial to focus on the wedge-shaped (z_h, P_{hT}) region, as a simple z_h cut proves insufficient.

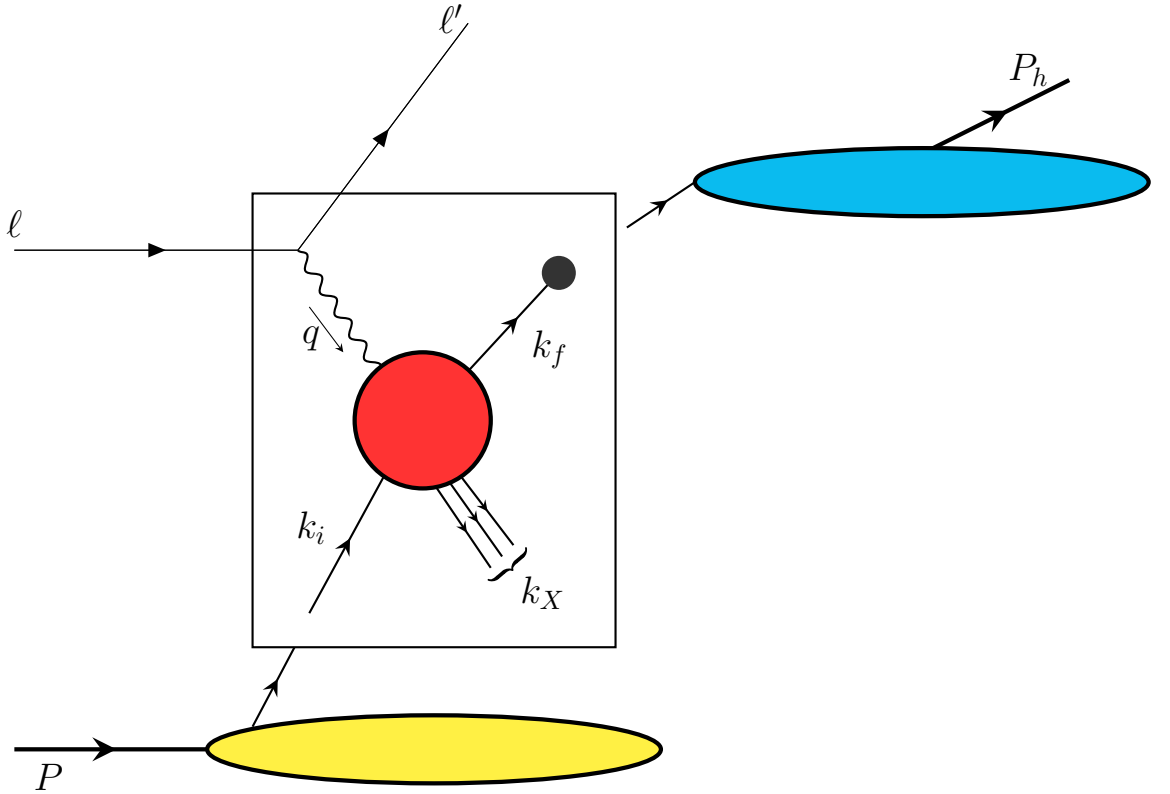


FIGURE 4.2: SIDIS sketch for Current Fragmentation Region with detailed internal process

Fig. 4.2 pictures the process where the initial hadron P and its remnants form a yellow blob at the bottom, while the detected hadron P_h emerges from a final-state blue blob at the top. Lines that are connecting

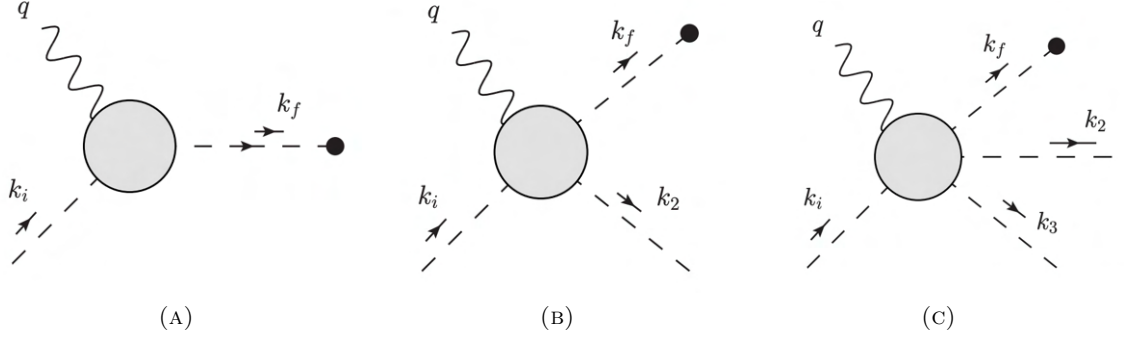


FIGURE 4.3: Examples of hard-scattering kinematics. Graph (A) representing the simplest partonic configuration $2 \rightarrow 1$, so-called Handbag kinematics. Graph (B) is $2 \rightarrow 2$ kinematics, which becomes relevant for massless pQCD calculations at large transverse momentum, typically providing the first non-zero contribution. Graph (C) represents $2 \rightarrow 3$ kinematics, describing more complex partonic interactions. In all cases (A) - (C), the dashed lines may correspond to composite entities, including gauge link structures or groups of particles.

these blobs trace the flow of momentum. It is important to mention that those lines are not just single quarks or gluons. They could represent entire groups of particles, like the collective effect of a gauge link or jet-like configurations. Importantly, we treat all these momenta as exactly known - with approximations. While the term "parton" usually brings to mind a single, massless, on-shell particle, we will keep using it for these dashed lines to avoid overly cumbersome language.

Now, focus on the partonic subprocess highlighted in the boxed region of Fig. 4.2. A black dot marks the specific parton linked to the observed hadron. The incoming struck parton carries momentum k_i , while at least one parton (with momentum k_f) hadronizes into the final-state hadron. All other unobserved partons contribute collectively, their total momentum labeled k_X .

For different specific partonic processes further approximations can be applied:

- First, in the $2 \rightarrow 1$ process of Fig. 4.3a, $k_i \rightarrow k$ ($k \equiv k_f - q$), and the $1/Q^2$ -suppressed terms are dropped.
- For a hard $2 \rightarrow 2$ process shown in Fig. 4.3b, $|k^2| \sim Q^2$ while $k_X^2/Q^2 \rightarrow 0$.
- If both $|k^2|$ and k_X^2 are large, then at least three partons (e.g., Fig. 4.3c) are ejected at wide angles from the hard collision.

For fixed x_N , z_N , Q^2 , and $\mathbf{P}_{\text{hT}}$, only certain k_i and k_f are consistent with any given picture in Fig. 4.3.

Although the diagram's layout naturally suggests a current-region interpretation, the same logic applies symmetrically to the target region. This perspective on factorization - where we systematically separate short-distance partonic physics from long-distance hadronic effects - builds on the foundational work of Collins and collaborators [4, 112].

4.2 Region indicators

To systematically identify **Current** (where the hadron emerges from the struck quark), **Target** (where it comes from the remnants of the original hadron), and **Central** (where dynamics are governed by soft processes) **fragmentation regions**, we introduce region indicators - ratios of partonic to hadronic momenta - first introduced in [95, 98]. These ratios help quantify the dominance of different kinematic regimes.

To understand when the process can be described in terms of partonic degrees of freedom, we define the *general hardness* ratio R_0 . It compares

key partonic momentum scales to the hard scattering scale Q^2 , taking the form:

$$R_0 \equiv \max \left(\left| \frac{k_i^2}{Q^2} \right|, \left| \frac{k_f^2}{Q^2} \right|, \left| \frac{\delta k_T^2}{Q^2} \right| \right). \quad (4.2)$$

Here, parameter δk_T^2 characterizes the parton's intrinsic transverse momentum in the large Q limit. The size of this momentum is equal to $\mathcal{O}(m^2)$, where m is the typical hadronic mass scale of the reaction. The partonic description of SIDIS processes becomes applicable when the general hardness ratio satisfies $R_0 \ll 1$, as established in [95, 98].

To separate the current fragmentation region from target and central regions, we introduce the *collinearity ratio* [98], R_1 , by

$$R_1 \equiv \frac{P_h \cdot k_f}{P_h \cdot k_i}. \quad (4.3)$$

For target and central fragmentation the value of collinearity must be large and it must be small for current fragmentation. The target region can be specifically identified using the *target proximity* ratio R'_1

$$R'_1 \equiv \frac{P_h \cdot P}{Q^2}, \quad (4.4)$$

which for target fragmentation [95, 98] is expected to be small.

The analysis of the TMD region can be explored by introducing *transverse hardness* ratio, defined as [5]

$$R_2 \equiv \frac{|k^2|}{Q^2}, \quad (4.5)$$

Since in the TMD regime the dominant process is the $2 \rightarrow 1$ scattering $\gamma q \rightarrow q'$ (characterized by $|k^2|/Q^2 \simeq 0$), the expression for R_2 is relevant.

To characterize the large transverse momentum region, we introduce the large transverse momentum ratio R_4 , defined in analogy with the general hardness ratio R_0 (Eq. 4.2). This ratio helps identify kinematic regimes where transverse momentum is generated primarily through perturbative QCD mechanisms. The explicit form of R_4 is:

$$R_4 \equiv \max \left(\left| \frac{k_i^2}{k^2} \right|, \left| \frac{k_f^2}{k^2} \right|, \left| \frac{\delta k_T^2}{k^2} \right|, \left| \frac{k_{iT}^2}{k^2} \right| \right), \quad (4.6)$$

It can be rewritten in terms of the transverse hardness ratio R_2 (Eq. 4.5) as:

$$R_4 = \frac{1}{R_2} \max \left(\left| \frac{k_i^2}{Q^2} \right|, \left| \frac{k_f^2}{Q^2} \right|, \left| \frac{\delta k_T^2}{Q^2} \right|, \left| \frac{k_{iT}^2}{Q^2} \right| \right), \quad (4.7)$$

A key feature of R_4 is its diagnostic power: when $R_4 \ll 1$, the observed transverse momentum arises significantly from hard perturbative processes, such as gluon radiation or high-momentum transfers. This makes R_4 particularly useful for describing the collinear QCD current region [113].

To investigate gluon radiation dynamics more thoroughly, we introduce the spectator virtuality ratio:

$$R_3 \equiv \frac{|k_X^2|}{Q^2}, \quad (4.8)$$

where $k_X = k_i + q - k_f$. This ratio provides crucial insight into the complexity of the partonic final state. When R_3 is small, the corresponding region can be explained by low-order (LO) perturbative (pQCD) calculations and it corresponds to $2 \rightarrow 2$ parton kinematics. On the other hand, if the process is $2 \rightarrow n$ scattering described by higher-order (HO) pQCD calculations, then both values R_3 and R_2 needed.

The transition region (which will be called *matching region* further), where R_2 takes intermediate values, is particularly interesting as it represents a kinematic domain where both TMD and collinear factorization approaches remain approximately valid. This overlap enables a smooth theoretical transition between these complementary frameworks, each providing valuable but distinct perspectives on the underlying QCD dynamics.

4.3 Affinity

To systematically evaluate how closely experimental data align with specific physical mechanisms across different kinematic regimes, we introduce the affinity tool $\mathcal{A}$. This Bayesian-inspired estimator provides a quantitative measure of proximity to theoretical predictions, serving as a global diagnostic for mechanism identification.

The affinity $\mathcal{A}$ is constructed as follows:

$$\begin{aligned} \mathcal{A}(x_{Bj}, z_h, Q^2, P_{hT} | \text{region}) &= \int d\{R_i\} \Theta(\{R_i\} | \text{region}) \int d^4k_i d^4k_f d^4\delta k_T \\ &\times \mathcal{P}(\{R_i\} | x_{Bj}, Q^2, z_h, P_{hT}; k_i, k_f, \delta k_T) \pi(k_i, k_f, \delta k_T), \end{aligned} \quad (4.9)$$

Region	R_0	R_1	R'_1	R_2	R_3	R_4
TMD	small	small	$\times$	small	$\times$	$\times$
matching	small	small	$\times$	small	$\times$	$\times$
collinear	small	small	$\times$	large	small (LO pQCD) large (LO pQCD)	small
target	small	large	small	$\times$	$\times$	$\times$
central	small	not small	not small	small	$\times$	$\times$

TABLE 4.1: Examples of sizes of region indicator ratios corresponding to particular regions of SIDIS. The symbol " $\times$ " denotes "irrelevant or ill-defined".

Let's take a closer look at the expression for affinity. The second term is a sophisticated probability engine that takes all our kinematic indicators R_i , introduced in the previous section, and calculates how likely they are to appear together, given what we already know about the system. This engine accounts for our prior knowledge (represented by prior distribution π) by essentially averaging over all possible starting assumptions and through the conditional probability $\mathcal{P}$, it weighs how probable different combinations of our indicators are for the specific collision events we have observed.

The affinity $\mathcal{A}$ computes the probability that given kinematics belong to a specific region by: (1) integrating over all possible parton momenta consistent with observations, (2) evaluating the joint distribution of R_i indicators, and (3) applying the proximity function Θ to select the relevant R_i ranges from Table 4.1. This Bayesian approach provides continuous, uncertainty-aware region classification.

To implement Eq. 4.9 in practice, we employ Monte Carlo methods, sampling four-vectors of parton momenta. The initial and final parton

momenta, k_i and k_f , are expressed in light-front coordinates as:

$$k_i = (k_i^+, k_i^-, \mathbf{k}_{iT}), \quad k_f = (k_f^+, k_f^-, \mathbf{k}_{fT}) \quad (4.10)$$

Here, the intrinsic transverse momentum of the parton is denoted as $\delta k_T = (0, 0, \delta \mathbf{k}_T)$. All the mentioned components are defined by the following formulas:

$$k_i^+ = \frac{Q}{\sqrt{2}\hat{x}_N}, \quad k_i^- = \frac{\hat{x}_N}{\sqrt{2}Q}(k_i^2 + k_{iT}^2), \quad \mathbf{k}_{iT} = (|\mathbf{k}_{iT}| \cos \varphi_{ki}, |\mathbf{k}_{iT}| \sin \varphi_{ki}) \quad (4.11)$$

$$k_f^+ = \frac{Q}{\sqrt{2}\hat{z}_N}(k_f^2 + (q_T \hat{z}_N \sin \varphi_h - |\delta \mathbf{k}_T| \sin \varphi_{\delta k})^2 + (q_T \hat{z}_N \cos \varphi_h - |\delta \mathbf{k}_T| \cos \varphi_{\delta k})^2),$$

$$k_f^- = \frac{Q}{\sqrt{2}\hat{z}_N}(k_f^2 + (q_T \hat{z}_N \sin \varphi_h - |\delta \mathbf{k}_T| \sin \varphi_{\delta k})^2 + (q_T \hat{z}_N \cos \varphi_h - |\delta \mathbf{k}_T| \cos \varphi_{\delta k})^2) \quad (4.12)$$

$$\delta \mathbf{k}_T = (|\delta \mathbf{k}_T| \cos \varphi_{\delta k}, |\delta \mathbf{k}_T| \sin \varphi_{\delta k}) \quad (4.13)$$

To properly implement this framework, we work with the Nachtmann scaling variable defined as:

$$x_N \equiv -q^+/P^+ = \frac{2x_{Bj}}{1 + \sqrt{1 + 4x_{Bj}^2 M^2/Q^2}} \quad (4.14)$$

which accounts for target mass effects, along with its rescaled counterpart $\hat{x}_N = x_N/\xi$ that relates the partonic momentum fraction to the hadronic scale. The parton kinematics are fully described through several key parameters: two fractions of the $0 < \xi, \zeta < 1$ representing the light-front momentum sharing; two invariant masses $k_i^2 < 0$ and $k_f^2 > 0$ (time-like, real final state), two transverse momenta $|\mathbf{k}_{iT}|, |\delta \mathbf{k}_T| > 0$ (initial

and intrinsic transverse momentum respectively) and two parton angular variables φ_{ki} and $\varphi_{\delta k}$ in the Breit system. The complete physical picture emerges when we include external kinematic variables x_N , z_N , q_T and φ_h .

Taking into account all these variables, we can rewrite equation (4.9):

$$\begin{aligned}
\mathcal{A}(x_{Bj}, Q^2, z_h, P_{hT} | \text{region}) &= \int d\{R_i\} \Theta(\{R_i\} | \text{region}) \\
&\times \int d\xi d\zeta dk_i^2 dk_f^2 d|\mathbf{k}_{iT}| d|\delta\mathbf{k}_T| d\varphi_{k_i} d\varphi_{\delta k} d\varphi_h \\
&\times \mathcal{P}(\{R_i\} | \xi, \zeta, k_i^2, k_f^2, |\mathbf{k}_{iT}|, |\delta\mathbf{k}_T|, \varphi_{k_i}, \varphi_{\delta k}) \\
&\times \pi(\xi, \zeta, k_i^2, k_f^2, |\mathbf{k}_{iT}|, |\delta\mathbf{k}_T|, \varphi_{k_i}, \varphi_{\delta k} | x_{Bj}, Q, z_h, q_T, \varphi_h). \quad (4.15)
\end{aligned}$$

While Equation 4.15 maintains generality through flexible choices of priors and proximity functions, we can focus our attention on three fundamental classes of priors for parton momenta: (1) momentum fractions, (2) angular variables, and (3) parton virtualities and intrinsic transverse momenta. These priors form the essential building blocks for characterizing the parton kinematics in our framework.

1. **Momentum fractions**. We employ flat priors within a window of size $\epsilon = 0.1$ around the physical values x_{Bj} and z_h :

$$\pi(\dots) \supset \theta(x_{Bj} < \xi < x_{Bj} + \epsilon) \theta(z_h < \zeta < z_h + \epsilon) \quad (4.16)$$

where θ denotes the Heaviside step function. This choice reflects the physical constraints $x_{Bj} < \xi < 1$ and $z_h < \zeta < 1$, with the lower-order diagrams satisfying $\xi = x_{Bj}$ and $\zeta = z_h$.

2. **Angular variables.** For the unpolarized SIDIS analysis, we adopt flat priors for all angular degrees of freedom:

$$\pi(\dots) \supset \theta(0 < \varphi_{k_i} < 2\pi) \theta(0 < \varphi_{\delta k} < 2\pi) \theta(0 < \varphi_h < 2\pi) \quad (4.17)$$

While angular modulations become non-trivial in spin-dependent cases, they remain simplified in our current framework.

3. **Parton virtualities and intrinsic transverse momenta.** The infrared dynamics of partons are modeled using Gaussian distributions:

$$\pi(\dots) \supset \mathcal{G}\left(\sqrt{|k_i^2|} \mid m, \Delta\right) \mathcal{G}\left(\sqrt{|k_f^2|} \mid m, \Delta\right) \mathcal{G}\left(|\delta \mathbf{k}_T| \mid m, \Delta\right) \mathcal{G}\left(|\mathbf{k}_{i,T}| \mid m, \Delta\right), \quad (4.18)$$

where $\mathcal{G}$ represents the Gaussian distribution with mean m and width Δ . This approach captures the expected behavior of parton virtualities and intrinsic transverse momenta in the non-perturbative regime.

For the Gaussian distributions in Eq. 4.18, we adopt the parameter values $m = 0.5$ GeV and $\Delta = 0.5$ GeV. These choices are motivated by nonperturbative QCD considerations, specifically through fits to the Di-jet Production Network (DPN) width from Refs. [114, 115]. The selected values are particularly appropriate for modeling the infrared dynamics relevant to the experimental measurements discussed in this work.

While more sophisticated distributions could be employed, we adopt these Gaussian priors to demonstrate the practical utility of the affinity framework through Eq. 4.15. To define the proximity function, we use a

simple plane equation:

$$\Theta(R_i|\text{region}) = \prod_i \theta(R_i < R_i^{\max}(\text{region}))\theta(R_i > R_i^{\min}(\text{region})) \quad (4.19)$$

where the product of Heaviside step functions θ isolates the kinematic region of interest by enforcing boundaries $R_i^{\min}$ and $R_i^{\max}$ for each variable R_i . The step functions in Eq. 4.19 serve as kinematic selectors, where first and second θ functions define the upper and lower boundaries respectively: $\theta(R_i < R_i^{\max})$ selects configurations below a maximum threshold and $\theta(R_i > R_i^{\min})$ retains configurations above a minimum threshold.

It is important to clarify that both boundaries are not always simultaneously required - the specific kinematic region determines which constraints are active. For example, in the case of a collinear TMD transition, we employ a modified selection $\theta(R_2^{\max}(\text{region}) < R_2 < R_2^{\min}(\text{region}))$. This inverted notation intentionally selects intermediate values between the specified boundaries, which is particularly relevant for transition regions between perturbative and non-perturbative dynamics and for kinematic configurations where neither extremely soft nor hard limits apply.

The flexibility in applying these constraints allows proper isolation of the relevant phase space for different physical regimes while maintaining mathematical rigor in the factorization procedure.

When working with transverse momentum dependent distributions, we face an important practical challenge: while Table 4.1 gives us reasonable estimates for the boundaries $R_{\min}$ and $R_{\max}$, we lack precise quantitative knowledge about where exactly these kinematic regions begin and end.

In this context, TMD phenomenology serves as a critical tool that gives us guidance in obtaining knowledge about the boundaries.

Imagine these boundaries as defining a three-dimensional "kinematic box" - a conceptual space containing all physically meaningful values for our analysis. To determine the size of this box, we turn to real experimental data. Specifically, we use measurements from Jefferson Laboratory's fixed-target experiments at 22 GeV, an energy scale that is particularly relevant for this study.

We study this process by running detailed Monte Carlo simulations - our main computational tool. These simulations carefully calculate all the important kinematic variables we need, follow how parton momenta change from the beginning to the end of the process.

The resulting three-dimensional parameter space, visualized in Fig. E.5, represents more than just abstract boundaries, it represents real physical behavior that emerges when theory meets experiment. This method keeps our work connected to both measurable results and solid TMD theory. The resulting approach makes theoretical sense while staying useful for real experiments.

The ratio of the number of events that fall within a defined kinematic region (the "box") to the total number of events is referred to as the *affinity*. This affinity, denoted as $\mathcal{A}$, provides insight into how well a particular kinematic configuration is populated. Following the approach in Ref. [116], kinematic regions where the data and phenomenology show

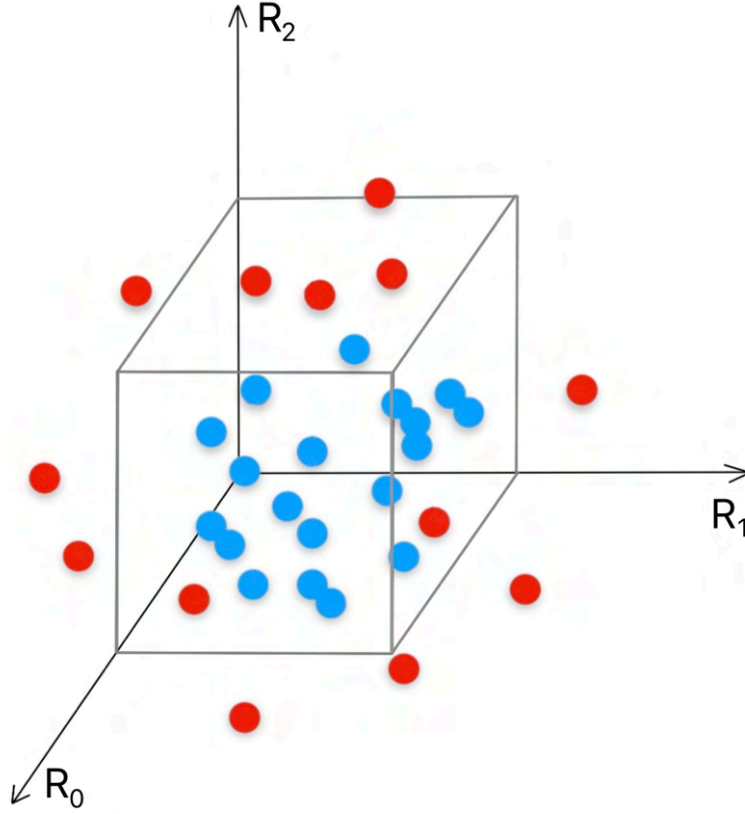


FIGURE 4.4: Formation of box and data based on TMD phenomenology. The blue dots indicate the data that fell inside the area of the box, the red dots indicate those that did not.

good agreement correspond to high affinity values ($\mathcal{A} \sim 1$), while excluded regions show low affinity ($\mathcal{A} \sim 0$). This principle guides the determination of allowed ranges for the parameters R_i : the closer the affinity is to 1, the more events populate the kinematic space defined by these parameters.

Affinity depends on the choice of the underlying functional form, which reflects initial beliefs about the possible ranges of the parton's initial momentum. To isolate specific regions, a flat distribution is used, with $R_i^{\min}$ and $R_i^{\max}$ defining the boundaries of the kinematic bins. These boundaries are chosen based on existing phenomenological studies. For the TMD region, "small" values are set as $R_i^{\min}(\text{TMD}) = 0.3$, and "large" values are

three times greater, giving $R_i^{\max}(\text{TMD}) = 0.9$, for $i = 0, 1, 2$. In the absence of dedicated studies for collinear, target, and central regions, the same boundaries are applied: $R_i^{\min}(\text{range}) = 0.3$ and $R_i^{\max}(\text{range}) = 0.9$. Further details on how $R_i^{\max}$ depends on the region-specific indicators R_i can be found in Ref. [113].

The values of $R_{\min}$ and $R_{\max}$ may vary depending on the kinematics, and in some cases, finer adjustments are necessary to properly separate the regions. To ensure clear separation of points in overlapping regions, we apply different scaling factors to q_T/Q : in the TMD region it is scaled by a factor of 8, in the collinear region by a factor of 4, and in the central region by a factor of 2.

All previous steps remain valid within this updated methodology. However, it is essential to highlight these modifications as they are critical to obtaining the results presented in this work.

4.3.1 Affinity training

The numerical evaluation of affinity is a computationally intensive and time-consuming task. To overcome this limitation, machine learning techniques were employed to train neural network models capable of fast and accurate affinity prediction.

The affinity data used for training the neural networks were generated by varying the maximum values of three key ratios, $R_0^{\max}$, $R_1^{\max}$, and $R_2^{\max}$, within the range $(0.05, 1.25)$, while keeping other non-perturbative parameters fixed. Approximately 15,000 unique parameter configurations

were generated, and for each configuration, the corresponding affinity values were computed.

The **TensorFlow** framework [117] was utilized to develop and train four separate neural network models, each dedicated to predicting the affinity corresponding to one of the following regions: TMD, collinear, target, and central. The networks are trained with seven input variables: x_{Bj} , z_h , P_{hT} , Q^2 , $R_0^{\max}$, $R_1^{\max}$, and $R_2^{\max}$. The process of selecting an optimal neural network architecture is inherently non-deterministic, requiring the adjustment of numerous hyperparameters, including the number and size of hidden layers, as well as the optimization algorithm employed during training.

To efficiently explore the hyperparameter space, the Hyperband search algorithm was implemented using the **KerasTuner** [118] package. The optimal architecture identified for each network consists of an input layer, two hidden layers, and an output layer, as illustrated in Figure 4.5. In this architecture, each layer is connected through learnable parameters (weights), updated iteratively during training based on provided input-affinity pairs.

The first hidden layer contains 576 neurons for the TMD network, 960 for the collinear, 896 for the target, and 576 for the central region. The second hidden layer comprises 160, 544, 256, and 736 neurons, respectively, for the same networks. All hidden layers employ the rectified linear unit (ReLU) activation function, defined as $f(x) = \max(0, x)$, to avoid the 'vanishing gradient' issue, where updates become too tiny to

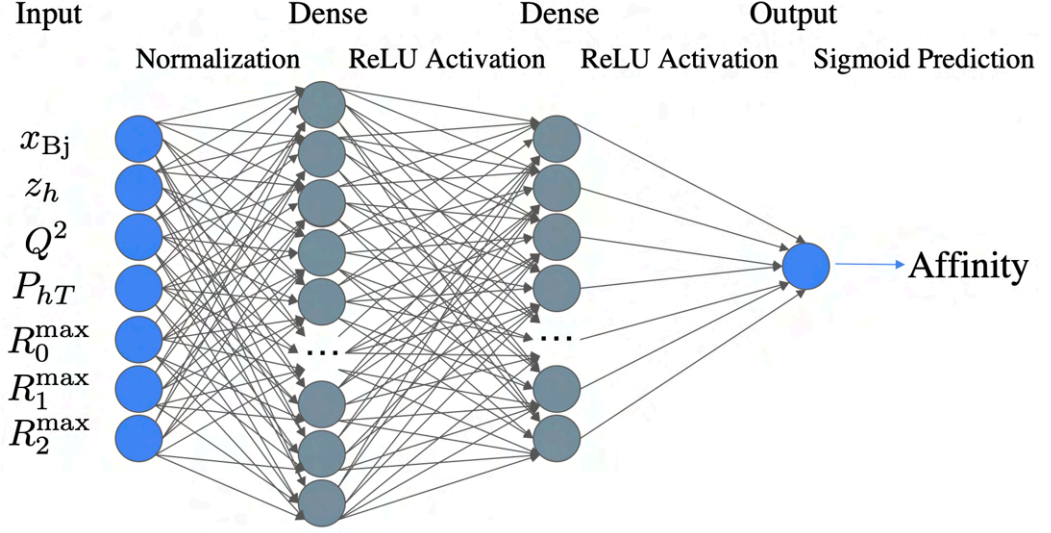


FIGURE 4.5: Schematic architecture of the neural network designed for affinity prediction. Blue blobs represent input and output layers, and gray blobs correspond to the hidden layers. Arrows are indicating trainable parameters. The model includes a normalization layer and ReLU activation functions to help prevent vanishing gradient issues.

help the model learn properly. The output layer utilizes a sigmoid activation function to constrain affinity predictions to the interval $(0, 1)$, ensuring compatibility with the probabilistic interpretation of the predicted values. Alternative output activations were considered during the tuning process but were found to be not optimal.

The dataset is partitioned into 80% for training and 20% for validation. During training, the Adam optimization algorithm is applied to minimize the mean squared error (MSE) between predicted and true affinity values on the validation set, thereby avoiding overfitting and ensuring generalization. Training proceeds for 50 to 500 epochs, depending on convergence behavior. For each hyperparameter search, the five models achieving the lowest validation MSE are retained. In cases where the best model choice was ambiguous, an independent test dataset was used to inform the final model selection for implementation in the affinity prediction framework.

All trained models are available in a public GitHub repository [119].

To further assess model performance, 100 independent datasets were generated for each experimental bin. These datasets enabled a systematic comparison between generated and predicted affinity values. It has been shown in [113] that residuals for the TMD network exhibit greater variance at higher affinity values, whereas residuals for the collinear network display more uniform behavior across the full affinity range. The standard deviations of the residuals, measured at 0.004 for the TMD affinity and 0.007 for the collinear affinity, confirm the high accuracy and reliability of the trained neural network models for rapid affinity evaluation.

4.4 Upgrades

In the work [113], the affinity for the current fragmentation region for unpolarized multiplicities was estimated using datasets from HERMES and COMPASS, as well as for the Jefferson Lab kinematics at 12 GeV and the future Electron-Ion Collider (EIC). The proximity to the current fragmentation region was also quantified for events characterized by large transverse momenta, which are typically described by collinear QCD approaches.

Lower-energy experiments such as those at Jefferson Lab exhibit a non-negligible admixture of central and target fragmentation events, complicating the interpretation of observed hadrons solely within the current fragmentation framework.

Affinity tool helped to explore the transition region between TMD and collinear factorization descriptions, providing insight into the boundaries where these theoretical frameworks overlap or break down. This led to further development of the tool and its application to energy-updated data at 22 GeV.

4.4.1 Insights on the exploring regions for affinity application

The studying of the kinematic reach of the experiments in terms of the produced hadron rapidity, defined as

$$y_h \equiv \frac{1}{2} \ln \left| \frac{P_h^+}{P_h^-} \right| \quad (4.20)$$

is relevant since rapidity is a physically meaningful observable. But it is difficult to measure, especially at large rapidity values where particle trajectories run close to the beam pipe, making precise determination of their energies and total momenta challenging. In experimental practice, pseudorapidity is frequently used instead. It depends only on the polar angle of the particle's trajectory relative to the beam axis, making it well suited for describing the detector's acceptance and the positioning of its various components. For highly relativistic particles, rapidity and pseudorapidity are nearly equivalent, allowing both to be used interchangeably in physics discussions.

The initial proton always lies in the positive rapidity region, while the produced hadron can have either positive or negative rapidity. Hadrons produced at negative rapidity are typically associated with the current

fragmentation region. Hadrons with larger transverse momentum (q_T) tend to shift toward central or positive rapidity and are likely to originate from hard gluon scattering processes, which can be described using collinear QCD. The central rapidity region, where low-energy partons undergo hadronization, generally corresponds to intermediate rapidity values. Lastly, the target fragmentation region is usually associated with hadrons produced at positive rapidities.

The region we are going to explore using affinity tool are: Target and Central regions, TMD and Collinear regions, and the most fascinating TMD-Collinear Matching region.

TMD region is a region where the transverse momentum dependent factorization is valid. The concept of TMD affinity gives us a quantitative measure of how given kinematic bin falls within this region. This quantity is calculated using Eq. (4.9) by requiring that the region indicators R_0 , R_1 , and R_2 to be sufficiently small (typically $R_{0,1,2} < 0.3$) to ensure that the theoretical framework remains applicable.

One of the most distinctive features of high TMD affinity regions is their kinematic sensitivity. Bins with a TMD affinity of 68% or higher are often associated with two key conditions: a small ratio of transverse momentum to the hard scale, $q_T/Q \lesssim 0.4$, and negative hadron rapidity, y_h , which indicates that the hadron is produced in the backward (current fragmentation) region. Together, these conditions ensure that the transverse momentum q_T remains perturbatively small, while the dynamics relevant to TMD factorization are preserved.

The interesting observation in [95] about a strong anti-correlation between the hadron rapidity y_h and the transverse momentum ratio q_t/Q was made. In other words, more negative rapidity values correspond to smaller q_t/Q ratios, reinforcing the dominance of TMD effects in that region. This interplay highlights that reliable classification of TMD-valid regions requires considering both variables simultaneously. In the same work the comparison to traditional kinematic cuts was made. It showed that comparison to the outlined in Refs. [116, 120] cuts, the TMD affinity approach shows excellent agreement while common cuts based only on hadron transverse momentum P_{hT} can be misleading. It happens because such cuts often fail to account for the dependence on the hadron energy fraction z_h , which can disrupt the $q_T \ll Q$ condition necessary for TMD factorization.

In the **Collinear region** hard recoils dominate and sensitivity to transverse momentum largely vanishes. In this regime, the complexities of intrinsic k_T fade into the background, replaced by a picture where partons scatter off each other with high momentum transfer, producing hadrons in a simpler, collinear fashion. But this simplicity comes with its own set of challenges.

The collinear region reveals itself through two key signatures. First, the region indicators serve as reliable markers: the quantity R_2 grows large, while R_4 remains small, signaling the dominance of hard, collinear processes. Second, the kinematic behavior shifts — the transverse momentum ratio q_T/Q becomes the defining variable, reflecting the growing influence of perturbative hard scattering.

However, recent experimental results have exposed lingering puzzles in this supposedly well-understood regime. Data from COMPASS and HERMES, for instance, show significant discrepancies with theoretical predictions — sometimes deviating by as much as an order of magnitude. Even sophisticated next-to-next-to-leading order ($\mathcal{O}(\alpha_s^2)$) calculations fail to fully resolve these tensions. Similar inconsistencies emerge in Drell-Yan measurements ([121]), pointing to deeper, unresolved questions about the nature of this kinematic territory.

The **TMD-collinear** matching region refers to the range of transverse momenta where $\Lambda_{\text{QCD}} \ll q_T \ll Q$. This intermediate zone represents the kinematic window in which a smooth transition between the TMD and collinear frameworks is expected. Within this range, it is generally possible to describe the data using either the TMD formalism or collinear factorization, providing flexibility in theoretical interpretation.

Traditionally, this region is handled through the well-known $W + Y$ construction, introduced by Collins, Soper, and Sterman (CSS) [122]. This method combines the TMD and collinear contributions in a way that produces a continuous, reliable cross-section across a wide range of q_T , while maintaining theoretical control over the uncertainties. The very existence of this matching region is a key requirement for the internal consistency of the CSS formalism.

In practice, however, achieving a reliable matching between the TMD and collinear descriptions—particularly for Semi-Inclusive Deep Inelastic Scattering (SIDIS)—has proven to be highly non-trivial. This challenge

becomes especially pronounced for lower-energy experiments, where the separation between the different regimes is less distinct. As discussed in Refs. [98, 123, 124], various alternative matching procedures have been proposed to address these difficulties, highlighting the ongoing theoretical effort required to fully understand and model the transition between these two fundamental regimes.

Target region. Partons that do not interact with the virtual photon during a scattering event hadronize and predominantly continue moving in the direction of the initial nucleon. The resulting hadrons from this process, known as *target fragmentation*, are typically found at positive rapidity, close to the beam line. While detecting these hadrons experimentally is challenging due to their proximity to the beam and the difficulty of separating them from the nucleon remnants, understanding target fragmentation remains important from both phenomenological and theoretical perspectives.

Theoretical descriptions of these processes are commonly formulated in terms of fracture functions [104, 106, 109, 125]. These functions represent the conditional probability of producing a hadron from the nucleon remnant, where the hadron carries approximately $1 - x_B$ of the nucleon's momentum. For hadrons in this region, the usual definition of z_h — the energy fraction carried by the hadron — becomes ill-defined, as $z_h \approx 0$ in this case.

Fracture functions play a key role in describing processes such as diffractive hadroproduction, and efforts to develop consistent factorization frameworks for these processes have been ongoing [105]. Factorized expressions for *TMD fracture functions* were first conjectured in Ref. [109], with further developments on their evolution equations and a more rigorous treatment of factorization presented in Ref. [125]. Additionally, Ref. [109] outlines specific correlations that could be tested experimentally, offering avenues to probe these theoretical constructs in future measurements.

The final region we consider in this discussion is the **Central Region**. This zone, located in rapidity between the struck quark and the nucleon remnants, is populated with radiation essential for the neutralization of color charge and the production of colorless hadrons from colored partons. Beyond its role in hadronization, the central region is also central to theoretical studies of factorization. The presence of soft radiation in this region is particularly significant for establishing the validity of factorization, including the TMD factorization framework [4, 5].

While Monte Carlo event generators commonly used in experimental analyses already incorporate a model for central fragmentation, this region has yet to be systematically explored in experimental studies. Future measurements, such as detailed studies of hadron rapidity distributions, will be crucial to improve our understanding of the central region and to validate the underlying theoretical assumptions.

4.4.2 Sensitivity of Affinity to Non-Perturbative Parameters

To ensure that affinity-based classifications are a reliable tool for guiding experimental studies and phenomenological investigations of nucleon structure, it is essential to assess how sensitive the affinity results are to the choice of non-perturbative parameters. In particular, identifying dominant parameters that significantly influence the affinity outcomes can improve both the interpretation of current results and the design of future experiments.

The region indicators R_i , which determine the affinity, depend on both measurable SIDIS observables and non-perturbative QCD parameters that are not precisely known. In this study, we consider four such non-perturbative parameters, each sampled from a random normal distribution within the range $(0, M_P)$, where M_P is the proton mass scale. To systematically explore the impact of these parameters on the affinity results, two complementary approaches were implemented:

- **Parameter Fixing:** The value of each parameter was fixed in turn to isolate its effect on the affinity outcomes. This method allows us to identify which parameter has the most pronounced influence on the affinity classification.
- **Lower Boundary Variation:** The lower bound of the output interval for a selected parameter was varied to test hypotheses regarding its dominant role in shaping the affinity results.

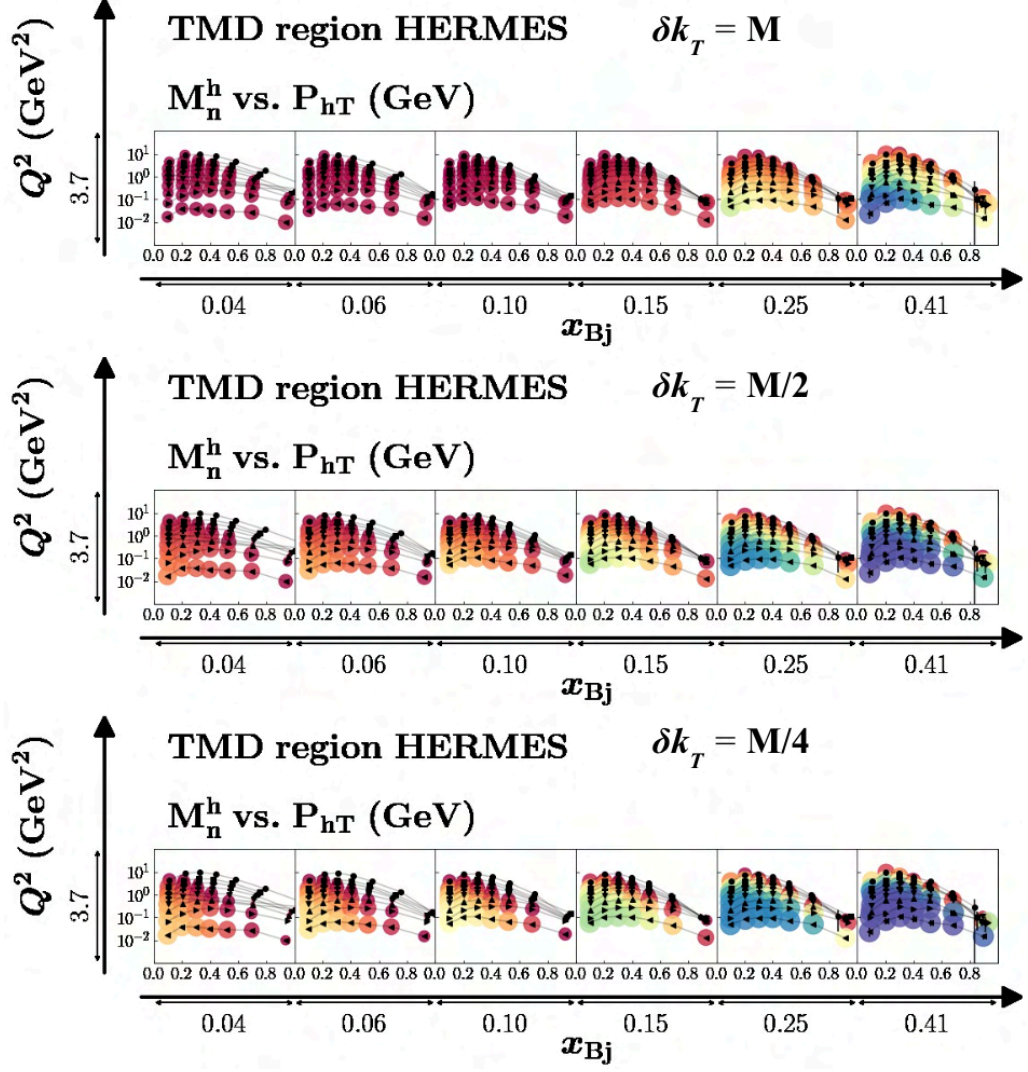


FIGURE 4.6: Affinity for HERMES data with δk_T parameter changing

These studies were conducted using newly developed software tools based on Jupyter Notebooks, which allowed for efficient and interactive exploration of the parameter space.

The analysis, which can be seen on Fig. 4.6, 4.7, revealed that among the four non-perturbative parameters, one parameter — the intrinsic transverse momentum of the parton δk_T — exhibits the strongest influence on the affinity results [126]. This parameter is particularly relevant as it effectively characterizes the intrinsic relative transverse momentum of partons, especially in the large- Q limit. The strong sensitivity of the

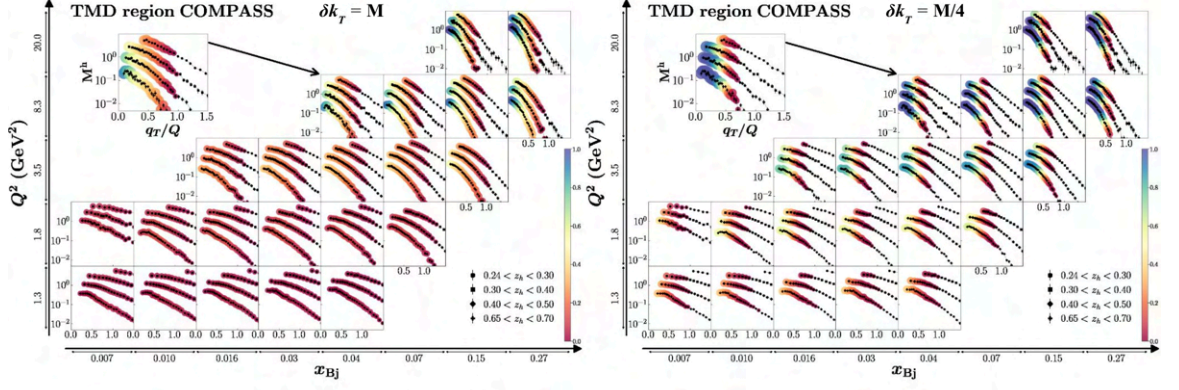


FIGURE 4.7: Affinity for COMPASS data with δk_T parameter changing

affinity outcomes to variations in δk_T underscores its importance for both theoretical modeling and experimental interpretation.

Furthermore, a notable feature emerged when comparing the affinity distributions for the TMD and collinear regions. The kinematic areas showing the highest affinity for the TMD region correspond to areas with the lowest affinity in the collinear regime. This clear separation reinforces the physical interpretation of these regions as distinct, with minimal overlap. Such behavior provides further validation of the affinity approach as a robust tool for distinguishing between the TMD and collinear regimes based on kinematic selections.

In summary, this study demonstrates that while affinity-based classifications are generally stable, they exhibit a pronounced dependence on the δk_T parameter [126]. This insight highlights the need for precise constraints on non-perturbative parameters to enhance the predictive power of affinity-guided studies in nucleon structure and QCD dynamics.

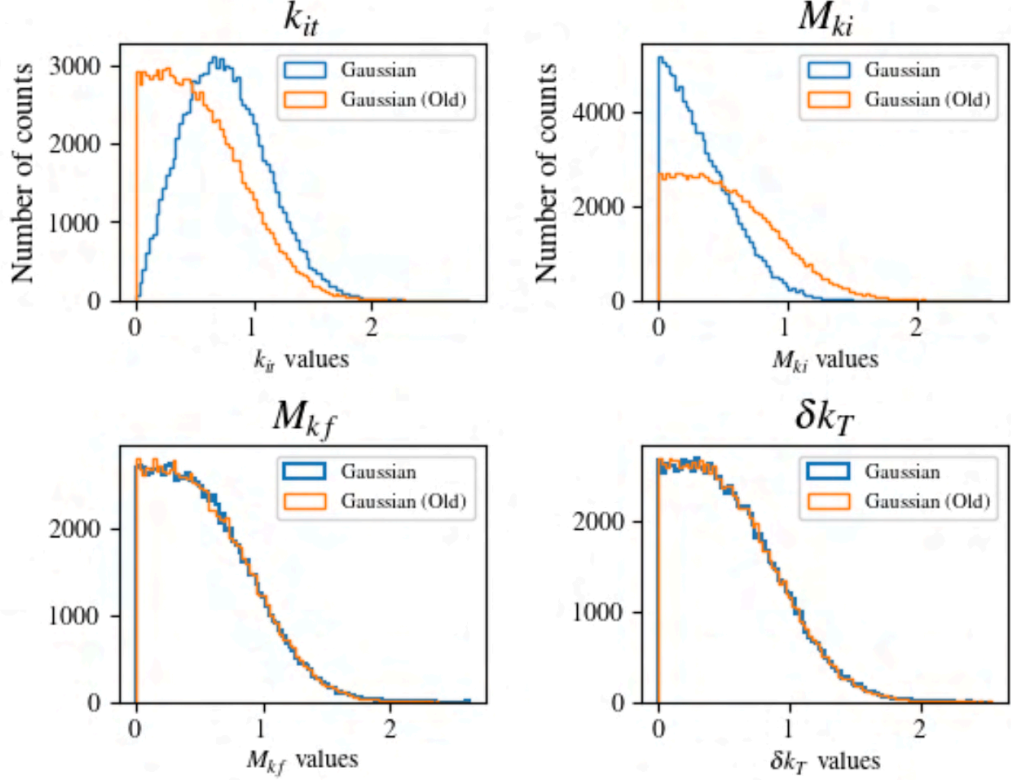


FIGURE 4.8: Distributions for non-perturbative parameters. Blue lines represent the new distribution with the implemented filter and orange lines are corresponding to pure Gaussian distributions

4.4.3 Exploring JLab data

In the absence of precise knowledge about the non-perturbative parameters entering the region indicator calculations, we initially adopted Gaussian distributions to model their behavior. While this provided a practical starting point, a closer inspection revealed that some of the generated parameter sets were not physically meaningful. Specifically, we identified that the condition $k_{iT}^2 - M_{ki}^2 > 0$ must always be satisfied to ensure physical reliability of the sampled parameter space [127].

To address this issue, a filtering procedure was introduced to exclude unphysical parameter combinations from the distributions. As a result, the shape of the parameter distributions changed as it shown in Fig. 4.8.

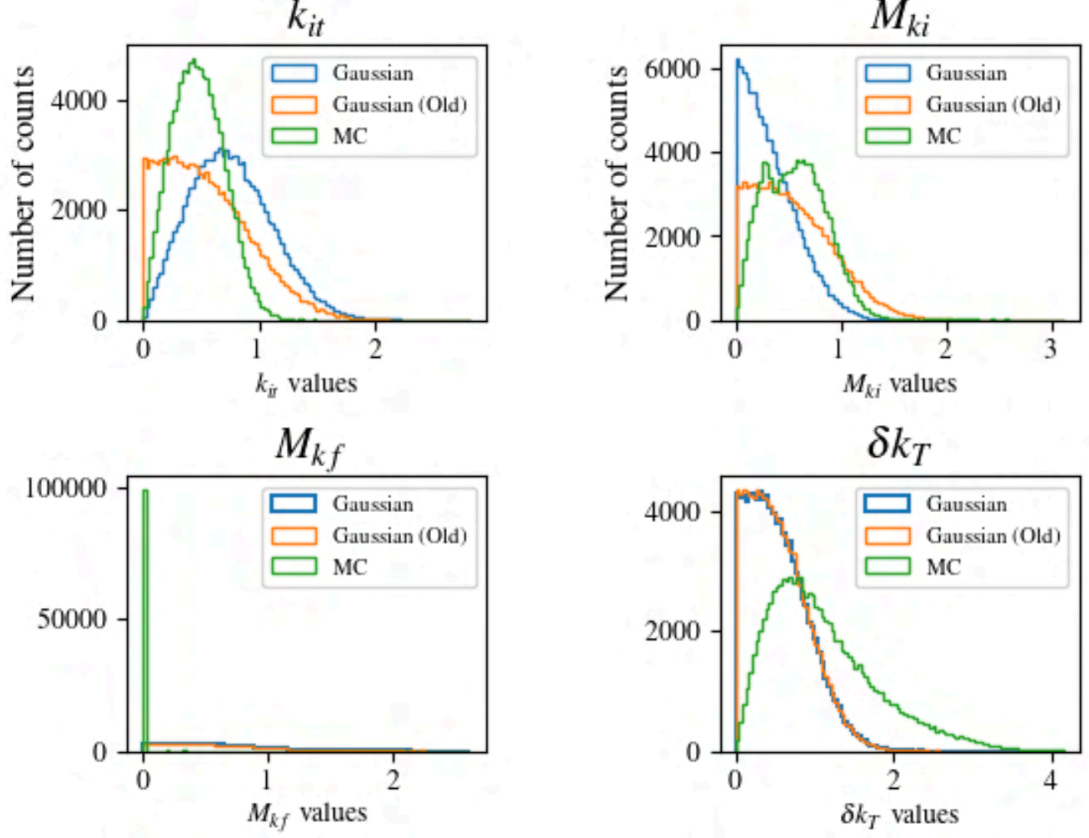


FIGURE 4.9: Distributions for non-perturbative parameters. Blue lines represent the new distribution with the implemented filter; orange lines are corresponding to pure Gaussian distributions; green lines represent the MC generated distributions

However, the distributions of the region indicators themselves remained largely unaffected by this modification, suggesting that the affinity outcomes were, at least initially, stable with respect to this refinement.

In the next step, we compared the filtered Gaussian-based distributions with parameter distributions generated through a more physically motivated Monte Carlo approach (Fig. 4.9). This comparison revealed significant differences between the two, not only in the distributions of the parameters themselves but also in the final affinity results derived from them (Fig. 4.10).

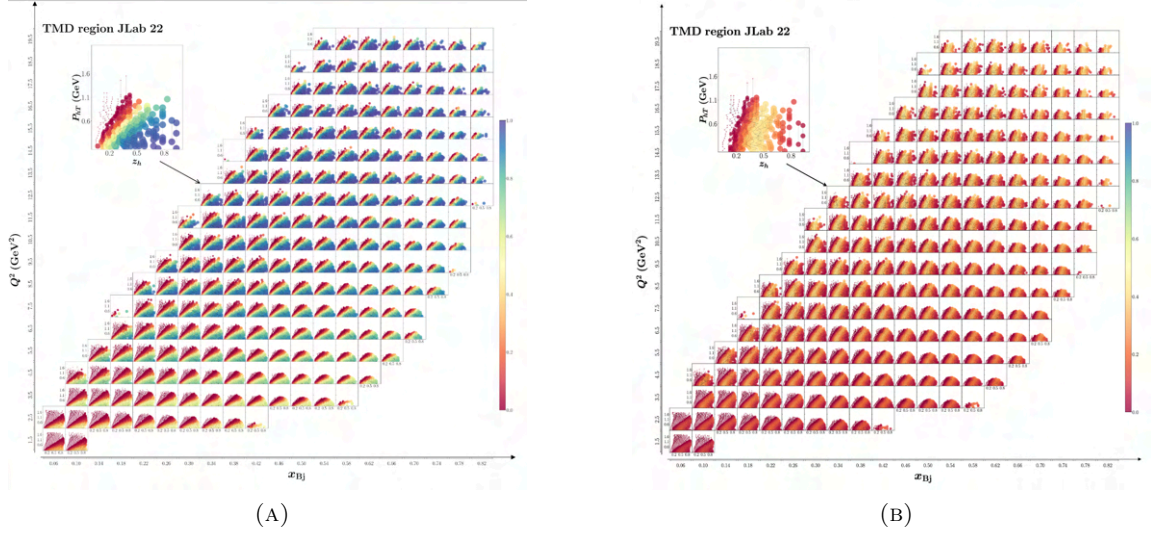


FIGURE 4.10: TMD affinity results using (A) Gaussian and (B) Monte-Carlo distributions for non-perturbative parameters

4.4.4 Changing the methodology of calculations

To investigate these differences in a concrete experimental context, we applied the Monte Carlo-based parameter distributions to the JLab12 kinematic dataset. This allowed us to systematically study how the choice of parameter modeling affects the affinity classifications while ensuring that important correlations between parameters are preserved throughout the analysis.

This investigation underscores the critical role of realistic parameter modeling in affinity-based studies. It also demonstrates that careful consideration of physical constraints and parameter correlations is essential to obtain reliable and interpretable results when studying the transition between TMD and collinear regimes. In the earlier affinity calculations, region indicators were determined as distributions for each bin value, and the mean of these distributions was assigned as the value of the region indicators. The dataset was binned such that each bin contained a similar

number of events, and kinematic values were represented by the central value of each bin. During the calculation of region indicators, all non-perturbative parameters - such as k_i, k_f, k_{it} , and δk_t - were modeled as Gaussian distributions, generated using a normal random function within the range from 0 to the M_h limit. The parameters ζ and χ were treated as random uniform distributions within the range observed in the Monte Carlo (MC) simulations for each bin, up to a maximum value of 1 [128].

Through the course of our research, we identified that the assumption of Gaussian distributions for the non-perturbative parameters was not valid, as it disregarded the inherent correlations among these parameters present in the simulation.

As a result of these findings, the methodology was revised to perform affinity calculations on an event-by-event basis, with data being binned only after the affinity values were determined, using the same distance-based approach. Affinity was calculated individually for each event, utilizing the actual non-perturbative parameters from the PYTHIA MC simulations. This ensured that the full correlation structure among the variables was preserved and that affinity values accurately reflected each specific generated event.

The affinity results obtained using the Monte Carlo distributions in event-by-event calculations demonstrated improved physical consistency. In particular, the rapidity-dependent plot (Fig. 4.11) clearly confirmed the expected kinematic coverage of the JLab12 experiment. As anticipated, the accessible phase space of JLab12 predominantly overlaps with

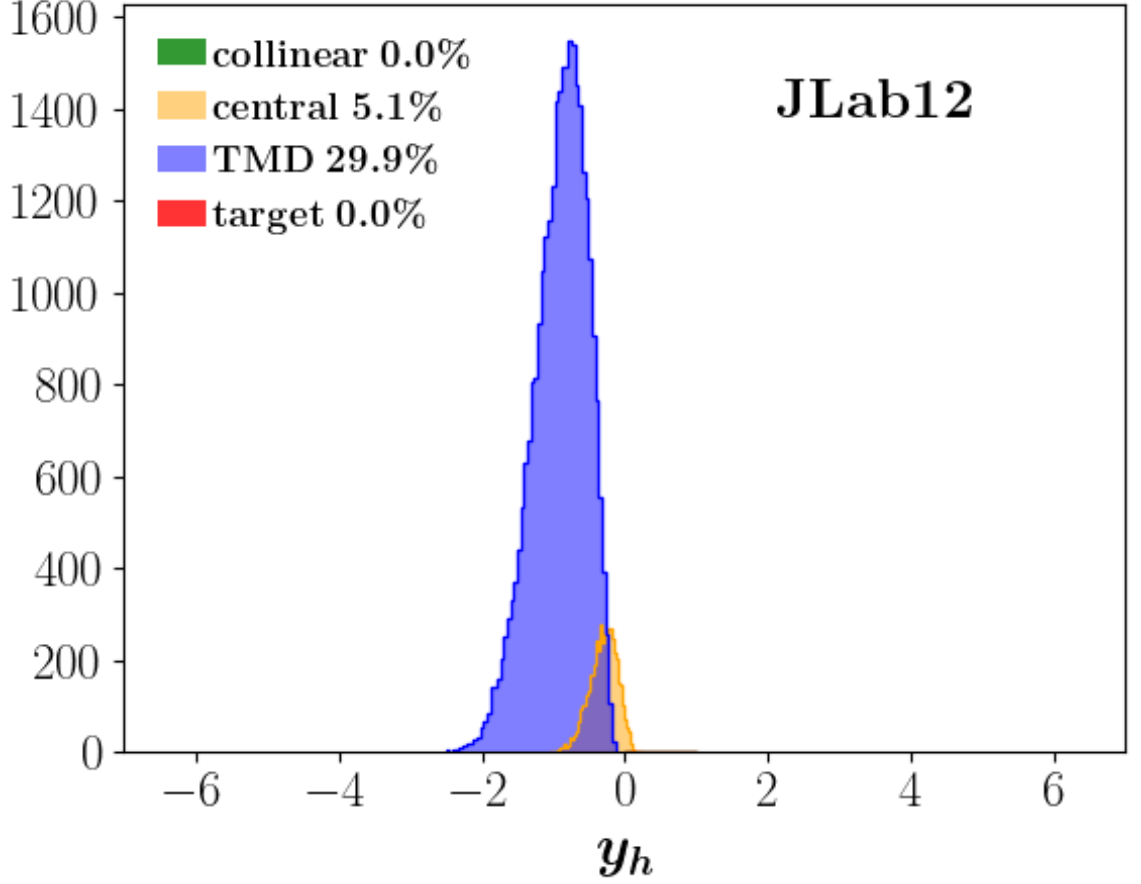


FIGURE 4.11: Rapidity for JLab12 dataset

the TMD region (Fig. 4.12), highlighting the experiment’s suitability for probing transverse momentum-dependent phenomena in nucleon structure.

After careful validation of the methodology, we applied the same event-by-event affinity calculation to the SIDIS datasets with π^+ and π^- outgoing hadrons. The rapidity plots changed a lot comparing to the previous method (Fig. 4.13) and showed us that collinear region will be reachable in 22 GeV update. The legends in Fig. 4.13 indicate the percentage of all kinematic bins with an affinity greater than 5%. For the JLab22 kinematics (experimental coverage presented in E.1), the TMD and collinear regions appear well separated, consistent with expectations.

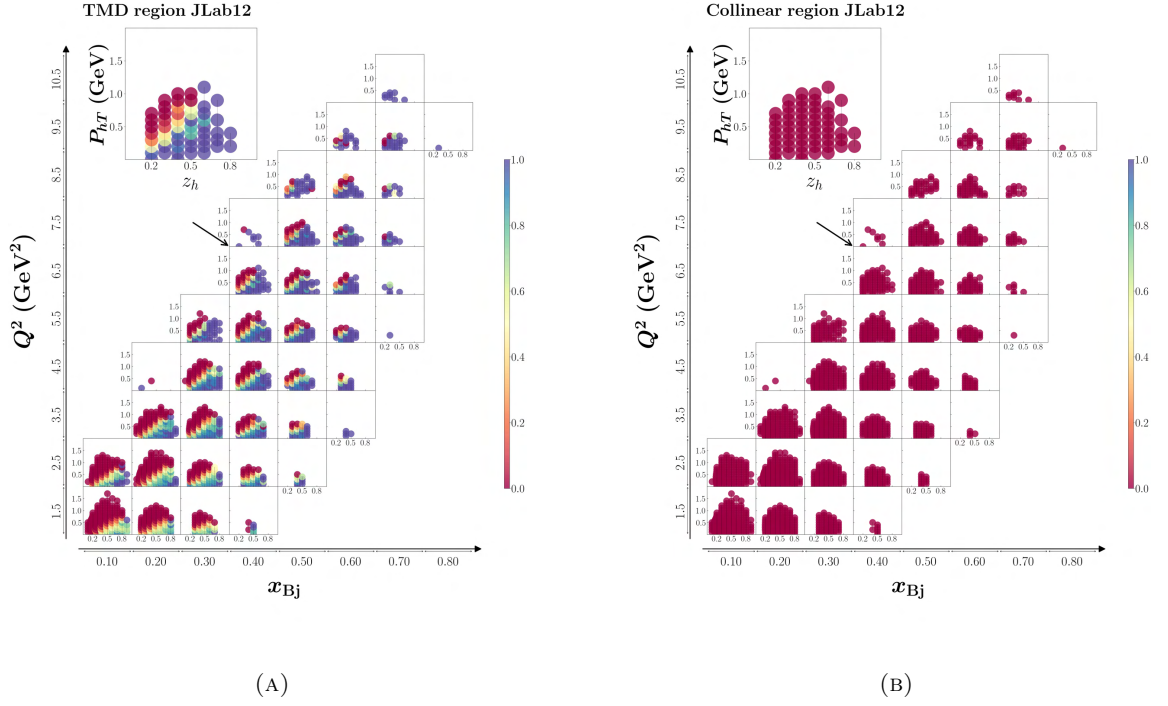


FIGURE 4.12: TMD affinity results for (A) TMD and (B) Collinear regions for JLab12 dataset π^+ hadron

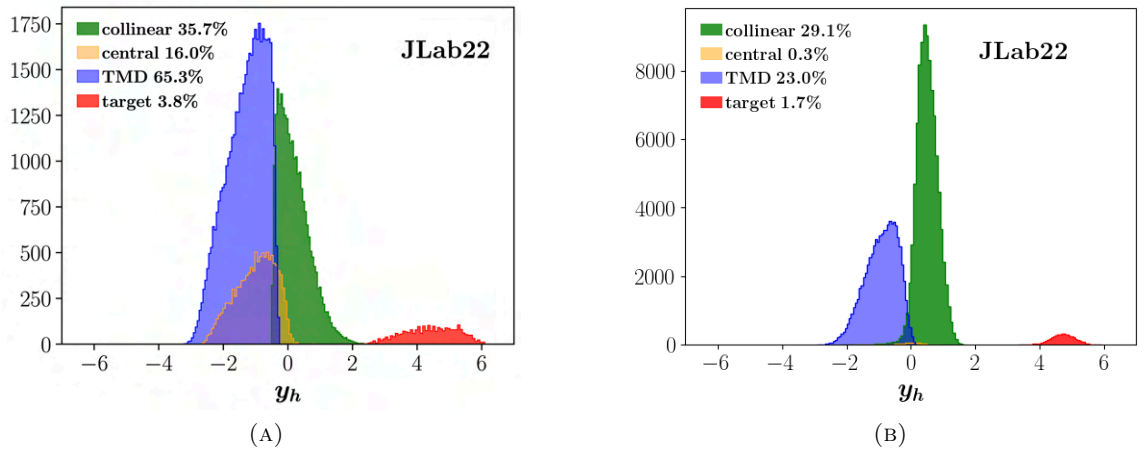


FIGURE 4.13: Rapidity phase space for JLab22 dataset: (A) using Gaussian distributions for non-perturbative parameters, (B) using event-by-event approach for the whole dataset

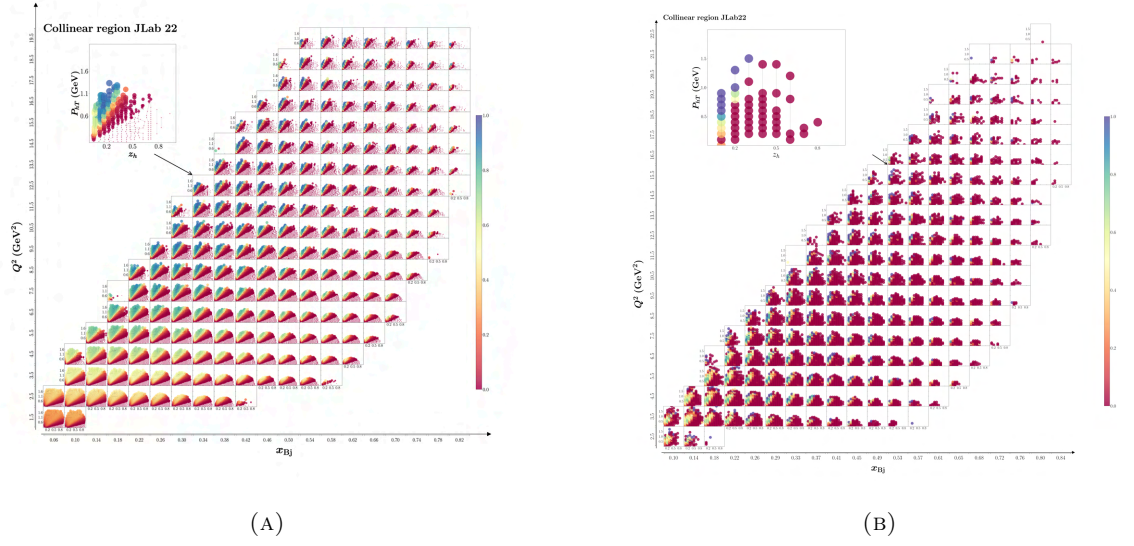


FIGURE 4.14: Affinity values for Collinear region: (A) using Gaussian distributions for non-perturbative parameters, (B) using event-by-event approach for the whole dataset

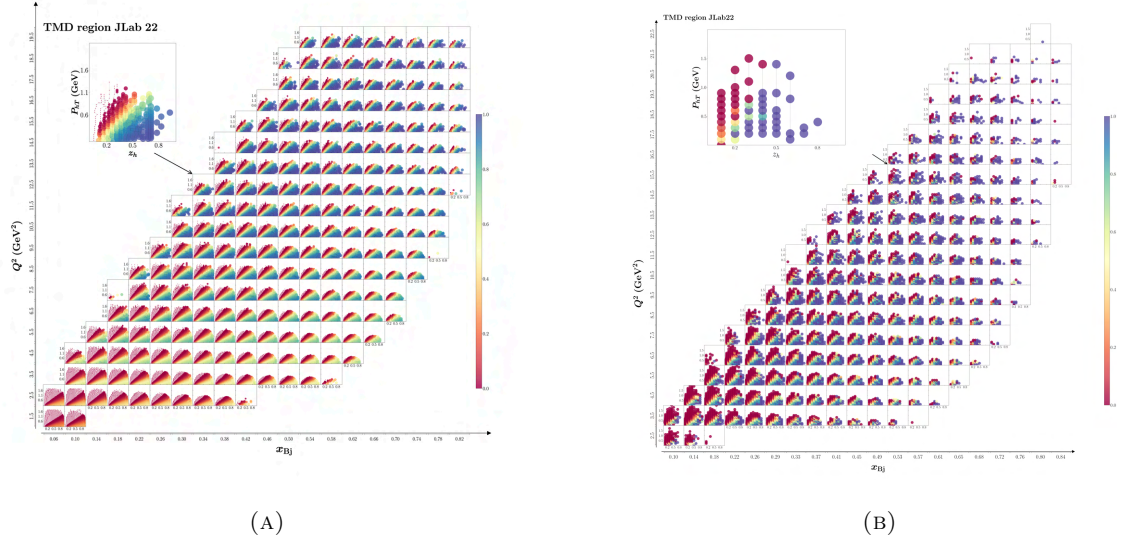


FIGURE 4.15: Affinity values for TMD region: (A) using Gaussian distributions for non-perturbative parameters, (B) using event-by-event approach for the whole dataset

The resulting affinity distributions are presented on Figures 4.14, 4.15 and 4.16, providing insight into the kinematic structure of these processes.

During the binning process, the affinity values of all events within a bin were averaged by dividing the sum of these values by the total number of events in the bin. This approach defines the final affinity value assigned

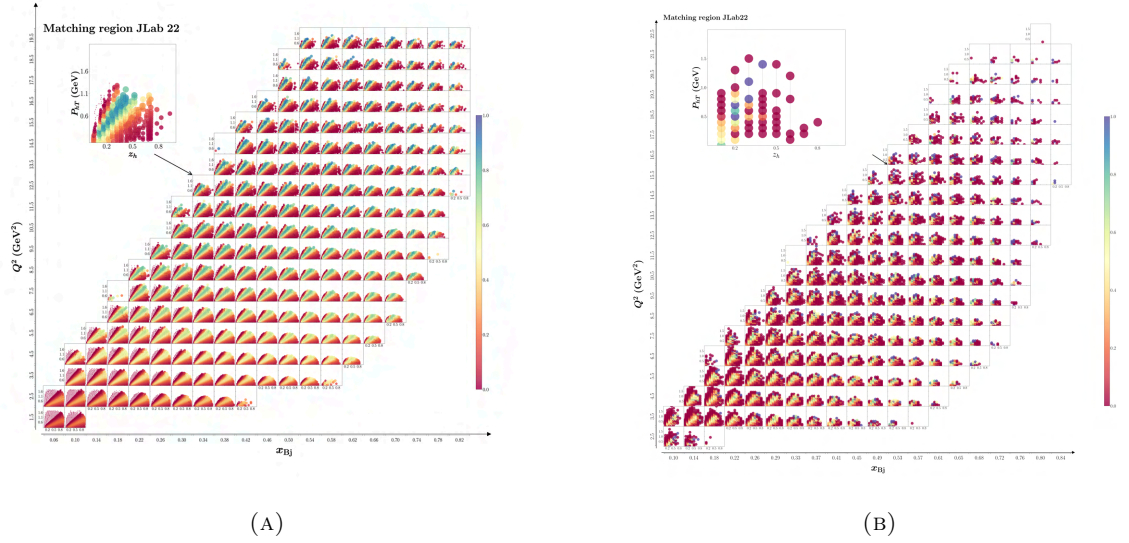


FIGURE 4.16: Affinity values for TMD-Collinear Matching region: (A) using Gaussian distributions for non-perturbative parameters, (B) using event-by-event approach for the whole dataset

to each bin.

Results demonstrate that the affinity, when calculated using the refined event-by-event methodology, exhibits significant differences compared to previous bin-averaged approaches. Notably, the TMD region is found to be considerably broader than previously estimated, while the collinear region appears narrower. This more precise picture is a direct consequence of the event-level resolution, which captures local kinematic fluctuations that are otherwise lost in bin-averaged analyses.

Perhaps the most intriguing outcome is the clear identification of the so-called *TMD-Collinear Matching region*. This intermediate region provides a valuable opportunity to investigate the kinematic boundaries where TMD factorization breaks down and collinear factorization becomes applicable. A detailed understanding of this transition region is essential for both theoretical developments and for optimizing kinematic

cuts in future experimental studies.

4.5 Summary

Phenomenological studies of TMD factorization have entered a high-precision era, resulting in significant advances in understanding the interplay between perturbative and non-perturbative QCD effects. Nevertheless, several open challenges remain, particularly in the practical implementation of factorization theorems:

- Non-perturbative model contributions inevitably leak into perturbative regions, complicating theoretical interpretations (e.g., the HSO problem).
- Analyses performed in b_T -space, while theoretically motivated, reduce phenomenological intuition. The Fourier transform involved requires integration over all b_T , making it difficult to disentangle contributions from small and large transverse distances.

Despite these challenges, simultaneous global fits of SIDIS, Drell-Yan, and e^+e^- annihilation data represent a powerful tool for probing TMD dynamics. However, to ensure reliable interpretation, such fits must be performed within consistent and theoretically solid frameworks.

In this context, a detailed kinematic assessment of SIDIS phase space is essential. It can identify energy and transverse momentum regions where the application of TMD factorization is most reliable and where transitions to collinear factorization occur. The affinity approach provides

a practical, data-driven method to distinguish between TMD, collinear, and matching regions, offering new insights into both factorization mechanisms and hadronization processes.

Chapter 5

Conclusions and outlook

This work has advanced the understanding of QCD dynamics by combining two complementary approaches: the theoretical modeling of multi-quark systems using the method of multiparticle fields, and the development of an affinity-based methodology to map the kinematic transition between transverse momentum dependent (TMD) and collinear factorization regimes in semi-inclusive deep inelastic scattering (SIDIS) processes.

Key Findings

Multi-Quark Systems: The multiparticle field approach has been demonstrated as a viable framework for describing multi-quark scattering processes. The model reproduces key qualitative features of experimental data, in particular the non-monotonic dependence of the proton elastic scattering cross section on the squared momentum transfer, t . These features arise naturally from spin effects within the model, even at the level

of simple loopless and basic loop diagrams. Furthermore, the application of the Laplace method for approximate evaluation of loop diagrams provides a practical computational tool, though further refinement is necessary to improve both accuracy and computational efficiency.

Affinity-Based Kinematic Mapping: The shift from bin-averaged to event-by-event affinity calculations represents a significant methodological improvement. This approach preserves the essential correlations among non-perturbative parameters, leading to more reliable and physically meaningful kinematic classification. The analysis revealed that the TMD region is substantially broader than previously estimated, particularly within the accessible kinematic range of the JLab12 experiment. Conversely, the collinear region appears narrower but becomes well accessible at JLab22 energies. Most notably, a distinct TMD-Collinear Matching Region has been identified, offering a unique opportunity to study the transition between factorization regimes.

Future Directions

To further advance these findings, several avenues for future research are proposed:

- **Theoretical Developments:** Extend the multiparticle field approach by including higher-order loop diagrams, with the goal of achieving quantitative agreement with experimental scattering data. Simultaneously, improve the Laplace method for loop calculations to enhance precision and reduce computational complexity.

- **Methodological Advancements:** Apply the affinity-based framework to other processes, such as Drell-Yan and e^+e^- annihilation, to enable unified kinematic mapping across different QCD processes. Integrate global fits with event-level parameter modeling to ensure consistent interpretation of SIDIS, Drell-Yan, and e^+e^- data.
- **Experimental Applications:** Utilize the identified TMD-Collinear Matching Region to optimize kinematic cuts for future experimental analyses. Collaborate with experimental groups to design measurements specifically targeting the transition region, thereby enhancing the precision of TMD and collinear extractions.

Broader Impact

By bridging theoretical models with data-driven affinity methods, this work provides a clearer roadmap for exploring both perturbative and non-perturbative aspects of QCD. The tools developed here, including spin-sensitive multi-quark models and correlation-preserving affinity analyses, will serve as essential components in future studies of nucleon structure and hard scattering processes.

Appendix A

Notations for SIDIS process

The Semi-Inclusive Deep Inelastic Scattering (SIDIS) process considered is:

$$\ell(l) + N(P) \rightarrow \ell'(l') + h(P_h) + X \quad (\text{A.1})$$

where $\ell(l)$ and $\ell'(l')$ correspond to incoming and scattered lepton with four-momentum l and l' respectively, $N(P)$ is target nucleon with four-momentum P , $h(P_h)$ is observed hadron with four-momentum P_h and X corresponds to unobserved final-state particles.

The final-state hadron is of type B . The symbol “ X ” indicates that the sum over final states includes all unobserved particles, including additional hadrons of type B . Each B hadron in an event is counted individually. A schematic illustration is shown in Figure [A.1](#)¹.

In this process, the proton carries momentum P , the virtual photon has momentum q , and the produced hadron carries momentum P_B . The incoming and scattered leptons have momenta l and l' , respectively. The

¹The azimuthal angles in this figure follow the Trento conventions [[129](#)].

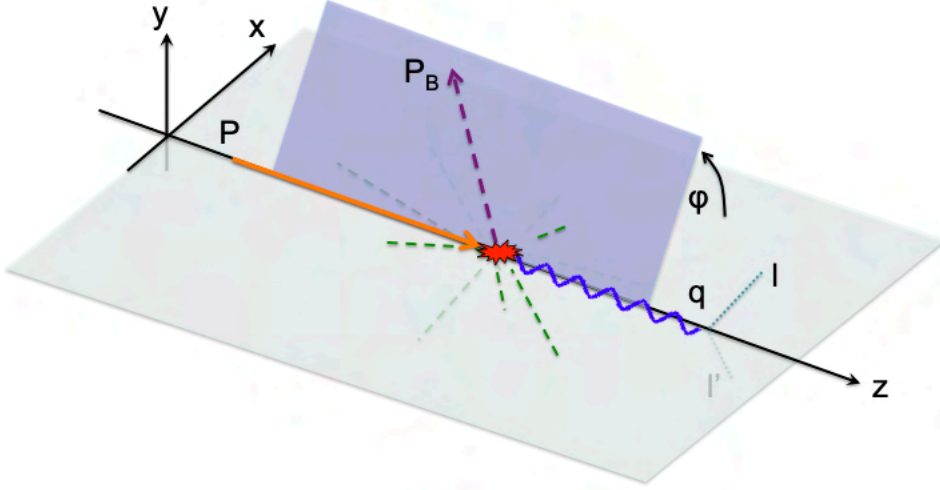


FIGURE A.1: Schematic representation of a SIDIS event in a photon frame (see Section 4.1). The hadron production plane is shown in purple. The dashed green lines represent unobserved final-state particles.

mass of the target proton is denoted by M , and the mass of the produced hadron is M_B .

Observables such as cross sections and structure functions are typically expressed in terms of the following kinematic variables:

$$Q^2 = -q^2 = -(l - l')^2, \quad x_{\text{Bj}} = \frac{Q^2}{2P \cdot q}, \quad (\text{A.2})$$

$$x_{\text{N}} = -\frac{q^+}{P^+} = \frac{2x_{\text{Bj}}}{1 + \sqrt{1 + \frac{4x_{\text{Bj}}^2 M^2}{Q^2}}}, \quad y = \frac{P \cdot q}{P \cdot l}, \quad (\text{A.3})$$

$$z_{\text{h}} = \frac{P \cdot P_B}{P \cdot q} = 2x_{\text{Bj}} \frac{P \cdot P_B}{Q^2}, \quad z_{\text{N}} = \frac{P_{\text{B}}^-}{q^-}, \quad (\text{A.4})$$

$$W_{\text{tot}}^2 = (q + P)^2, \quad W_{\text{SIDIS}}^2 = (q + P - P_B)^2, \quad s = (l + P)^2. \quad (\text{A.5})$$

The light-cone components $q^\pm$, P^+ , and P_{B}^- appearing in the definitions

of x_N and z_N are defined in a photon frame (see Appendix C), where the incoming proton moves along the positive z -axis with zero transverse momentum, and the virtual photon moves along the negative z -axis, also with no transverse momentum. Since longitudinal boosts along the z -axis do not affect these light-cone ratios, the precise choice of photon frame is unimportant for these definitions.

The variable x_N corresponds to the Nachtmann scaling variable, commonly denoted by ξ in the literature. To avoid confusion with the partonic momentum fraction ξ used elsewhere in this work, we use the symbol x_N , with subscripts distinguishing it from the Bjorken variable x_{Bj} .

For fragmentation descriptions, the light-cone fraction z_N serves as the analogue of x_N , and the subscript “N” emphasizes this analogy. The variables x_{Bj} , z_h , and P_B correspond to the x , z , and P_h definitions in Ref. [130]. Similarly, z_N corresponds to ζ_h as defined in Ref. [131].

A useful additional variable for describing the target fragmentation region is:

$$x_h = \frac{q \cdot P_B}{P \cdot q} . \quad (\text{A.6})$$

In the deep inelastic limit, we take $Q \rightarrow \infty$ while keeping x_N and z_N fixed, and treating $m/Q \rightarrow 0$, where m represents a generic mass scale much smaller than Q , such as a small hadron mass or Λ_{QCD} .

The kinematic variables satisfy:

$$Q^2 = x_{Bj}y(s - M^2 - m_l^2) \approx x_{Bj}ys , \quad (\text{A.7})$$

where m_l is the lepton mass, and the approximation symbol “ $\approx$ ” indicates that terms suppressed by powers of m/Q have been neglected while keeping x_N and z_N fixed.

Appendix B

Light-Cone Variables

Light-cone coordinates for a four-vector V^μ are defined as:

$$V^\mu = (V^+, V^-, \mathbf{V}_T), \quad (\text{B.1})$$

where the components are given by:

$$V^+ = \frac{V^0 + V^z}{\sqrt{2}}, \quad V^- = \frac{V^0 - V^z}{\sqrt{2}}, \quad \mathbf{V}_T = (V^x, V^y). \quad (\text{B.2})$$

For a four-momentum V , the rapidity is defined in the usual way:

$$y = \frac{1}{2} \ln \left| \frac{V^+}{V^-} \right|. \quad (\text{B.3})$$

In terms of rapidity, the four-momentum V can be expressed as:

$$V = \left(\frac{M_T}{\sqrt{2}} e^y, \frac{M_T}{\sqrt{2}} e^{-y}, \mathbf{V}_T \right), \quad (\text{B.4})$$

where $V^2 = M^2$, and the transverse mass is defined as:

$$M_{\text{T}} = \sqrt{|M^2 + \mathbf{V}_{\text{T}}^2|}. \quad (\text{B.5})$$

In the case of a virtual momentum with $M^2 < 0$, one of the light-cone components becomes negative. For example:

$$V = \left(\frac{M_{\text{T}}}{\sqrt{2}} e^y, -\frac{M_{\text{T}}}{\sqrt{2}} e^{-y}, \mathbf{V}_{\text{T}} \right). \quad (\text{B.6})$$

To specify components of four-momenta, we use the notation:

$$V_{\text{b},\text{c}}^{\text{a}}, \quad (\text{B.7})$$

where a indicates the Lorentz component (such as $+$, $-$, x , y , etc.), c specifies the reference frame and b is an optional subscript for additional context if required.

The transverse two-dimensional momentum in a given frame c is denoted by:

$$\mathbf{V}_{\text{b},\text{c},\text{T}}. \quad (\text{B.8})$$

The frame subscripts b and c consistently indicate in which reference frame the momentum components are expressed.

Appendix C

Reference frames

In the analysis of SIDIS kinematics, it is common and useful to work in different reference frames, particularly photon and hadron frames. Below, we outline the kinematic descriptions in these frames.

C.1 Photon Frame

In a photon frame, both the incoming proton and the virtual photon have zero transverse momentum, while the final-state hadron can carry non-zero transverse momentum. The relevant four-momenta are given by:

$$q_\gamma = \left(-x_N P_\gamma^+, \frac{Q^2}{2x_N P_\gamma^+}, \mathbf{0}_T \right), \quad (\text{C.1})$$

$$P_\gamma = \left(P_\gamma^+, \frac{M^2}{2P_\gamma^+}, \mathbf{0}_T \right), \quad (\text{C.2})$$

$$P_{B,\gamma} = \left(\frac{\mathbf{P}_{B,\gamma,T}^2 + M_B^2}{2P_{B,\gamma}^-}, P_{B,\gamma}^-, \mathbf{P}_{B,\gamma,T} \right). \quad (\text{C.3})$$

Here, the subscript γ indicates that all momentum components are expressed in the photon frame, following the convention in Equation ??.

The minus component of the final-state hadron momentum is:

$$P_{\text{B},\gamma}^- = \frac{z_{\text{h}}Q^2}{4x_{\text{Bj}}P_{\gamma}^+} \left(1 \pm \sqrt{1 - \frac{4x_{\text{Bj}}^2 M^2 (\mathbf{P}_{\text{B},\gamma,\text{T}}^2 + M_{\text{B}}^2)}{z_{\text{h}}^2 Q^4}} \right) \approx \frac{z_{\text{h}}Q^2}{2x_{\text{Bj}}P_{\gamma}^+}, \quad (\text{C.4})$$

where the approximation corresponds to neglecting hadron mass terms in the current fragmentation region.

In the photon frame, the azimuthal angles ψ and ϕ describe the orientation of the scattered lepton and produced hadron, respectively. Ratios involving $+$ and $-$ components are invariant under boosts along the z -axis, making them independent of the specific photon frame choice.

C.2 Breit Frame

A special case of the photon frame is the Breit frame (also known as the Brick Wall frame), illustrated in Figure C.1(a). In this frame:

$$q_{\text{b}} = \left(-\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \mathbf{0}_{\text{T}} \right), \quad (\text{C.5})$$

$$P_{\text{b}} = \left(\frac{Q}{x_{\text{N}}\sqrt{2}}, \frac{x_{\text{N}}M^2}{\sqrt{2}Q}, \mathbf{0}_{\text{T}} \right) = \left(\frac{M}{\sqrt{2}}e^{y_{P,\text{b}}}, \frac{M}{\sqrt{2}}e^{-y_{P,\text{b}}}, \mathbf{0}_{\text{T}} \right). \quad (\text{C.6})$$

The subscript b indicates quantities expressed in the Breit frame.

Throughout this work, unless otherwise stated, all four-momentum components are assumed to be in the Breit frame by default.

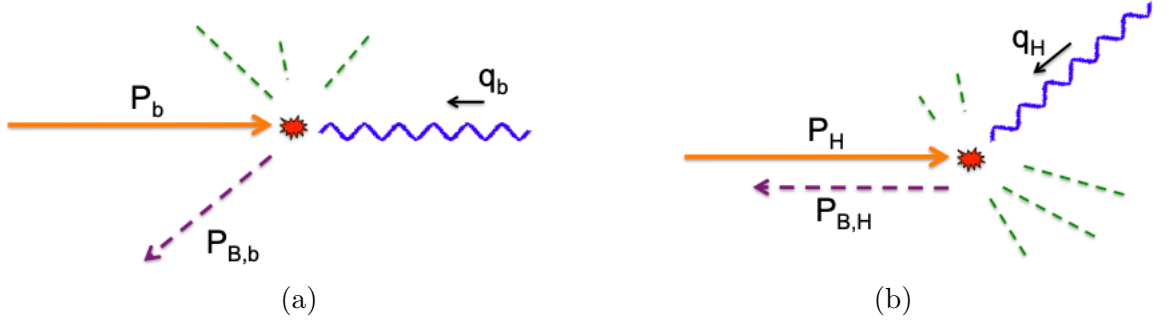


FIGURE C.1: Illustration of the kinematics in (a) the Breit photon frame and (b) the hadron frame. Dashed green lines represent unobserved particles.

This work is focusing on the Breit frame where the exchanged photon has vanishing energy and moves along the $-\hat{z}$ direction, while the target proton moves in the $+\hat{z}$ direction. The significance of the Breit frame is that in the limit of exactly collinear parton kinematics, the 3-momentum of the struck quark, initially in the $+\hat{z}$ direction, is exactly reversed in the hard collision.

C.3 Hadron Frame

The hadron frame, labeled by the subscript “H” and shown in Figure C.1(b), is defined such that the incoming and outgoing hadrons are exactly back-to-back, with zero relative transverse momentum. In contrast, the virtual photon generally carries non-zero transverse momentum in this frame. This configuration is especially useful for formulating factorization, as discussed in [4, Sec. 13.15.5].

The four-momenta in the hadron frame are:

$$q_{\text{H}} = (q_{\text{H}}^+, q_{\text{H}}^-, \mathbf{q}_{\text{HT}}), \quad (\text{C.7})$$

$$P_{\text{H}} = \left(P_{\text{H}}^+, \frac{M^2}{2P_{\text{H}}^+}, \mathbf{0}_{\text{T}} \right), \quad (\text{C.8})$$

$$P_{\text{B,H}} = \left(\frac{M_{\text{B}}^2}{2P_{\text{B,H}}^-}, P_{\text{B,H}}^-, \mathbf{0}_{\text{T}} \right). \quad (\text{C.9})$$

For definiteness, we choose the hadron frame such that the components of the incoming proton momentum match those in the Breit frame:

$$P_{\text{H}}^+ = P_{\gamma}^+ = \frac{Q}{\sqrt{2}x_{\text{N}}}, \quad (\text{C.10})$$

$$P_{\text{H}}^- = P_{\gamma}^- = \frac{x_{\text{N}}M^2}{\sqrt{2}Q}, \quad (\text{C.11})$$

$$\mathbf{P}_{\text{HT}} = \mathbf{0}_{\text{T}}. \quad (\text{C.12})$$

Appendix D

List of the applicant's publications on the topic of the dissertation and information on the approbation of the dissertation results

D.1 List of publications

1. Chudak N., Sharph I., Potiienko O., Milieva A., Merkotan K., Yushkevych T. and Rusov V., Laplace's method for elastic scattering diagrams within multiparticle fields model, Phys. Review D 106, 3 (2022) DOI:<https://doi.org/10.1103/PhysRevD.106.036023>
2. O. Potiienko, K. Merkotan, N. Chudak, D. Ptashynskiy, T. Zelensova, T. Yushkevich, I. Sharph, and V. Rusov, New method of accounting for interference contributions within a multiperipheral model, Physical Review D 101, 7 (2020) DOI:<https://doi.org/10.1103/PhysRevD.101.076021>

3. D.A. Ptashynskiy, T.M. Zelentsova, N.O. Chudak, K.K. Merkotan, O.S. Potiienko, V.V. Voitenko, O.D. Berezovskiy, V.V. Opyatyuk, O.V. Zharova, T.V. Yushkevich, I.V. Sharph, V.D. Rusov, Multiparticle fields on the subset of simultaneity, Ukrainian Journal of Physics 64(8), 732 (2019). DOI: <https://doi.org/10.15407/ujpe64.8.732>

D.2 Approbation of dissertation materials

1. T. Yushkevych, M. Boglione, A. Prokudin, "Affinity tool. Comparison between JLab12 and JLab22", "Science at the Luminosity Frontier: Jefferson Lab at 22 GeV" (Frascati, Italy, 2024) (<https://agenda.infn.it/event/39742/contributions/247698/>)
2. T. Yushkevych, I. Sharph, et.al, "Laplace method for single-loop diagrams of elastic proton scattering", 21-th Gamow International Astronomical Conference-School "Astronomy and beyond: astrophysics, cosmology and gravitation, high energy physics, astroparticle physics, radioastronomy and astrobiology", August 15-21, 2021 Odessa, Ukraine <https://gamow.odessa.ua/wp-content/uploads/2021/08/gamow-abstracts-2021.pdf>
3. T. Yushkevych, et.al, "Application of the Laplace method for one-loop diagrams of elastic scattering of protons in the framework of multiparticle fields", International Conference of Students and Young

Researchers in Theoretical and Experimental Physics “HEUREKA-2021”, May 18-20, 2021. Lviv, Ukraine <https://physics.lnu.edu.ua/conferences/heureka2021/>

4. Yushkevich T., Potiienko O., Neboga H., “Modeling the interference effects in elastic hadrons scattering with the Monte Carlo event generator based on Laplace’s method”, 20-th Gamow International Astronomical Conference-School “Astronomy and beyond: astrophysics, cosmology and gravitation, high energy physics, astroparticle physics, radioastronomy and astrobiology” 9–16 August, 2020, Odessa, Ukraine <http://gamow.odessa.ua/wp-content/uploads/2020/10/>
5. D.A. Ptashynskiy, T.V. Yushkevich, I.V. Sharph, V.D. Rusov, et.al, “Multi-particle fields on a subset of simultaneity”, New Trends in High-Energy Physics, May 12-18, 2019, Odessa, Ukraine. <https://indico.bitp.kiev.ua/event/1/contributions/62/>

In addition to the publications and conference abstracts listed above, one article has been uploaded to arXiv (Sharph I. V., Zelentsova T. M., Chudak N. O., Potiienko O. S., Ptashynskiy D. A., Merkotan K. K. Yushkevich T. V. and Milieva, A. O., Laplace method for the simplest diagrams of elastic scattering of scalar particles, eprint arXiv:2003.07321, <https://arxiv.org/abs/2003.07321>), and another article is currently being prepared for submission to a Scopus-indexed journal. The main results and conclusions of the dissertation have also been presented and discussed at the International School “Frontiers in Nuclear and Hadronic

Physics 2025” (Galileo Galilei Institute, Florence, Italy, 2025) (<https://www.ggi.infn.it/showevent.pl?id=494>), the 49th SLAC Summer Institute (California, USA, 2021), the “Amplitudes 2021” workshop (Niels Bohr International Academy and Niels Bohr Institute, Penn State, USA, 2021), IrisHEP Fellowship program (Princeton University, USA, 2022) and the International Conference of Students and Young Scientists in Theoretical and Experimental Physics “Eureka-2020” (Lviv, Ukraine, 2020).

Appendix E

Figures

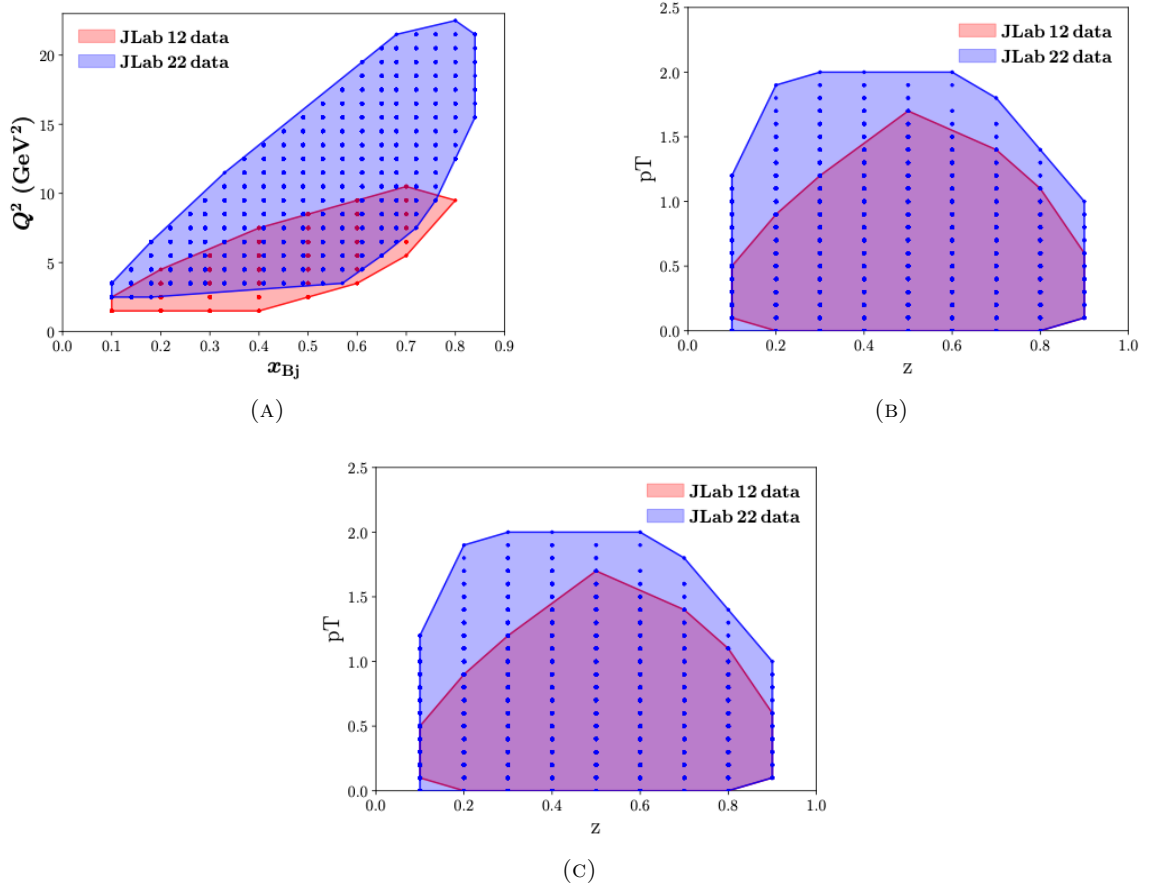


FIGURE E.1: Experimental coverage for JLab experiment at 12 and 22 GeV in: (A) Q^2 over x_{Bj} frame, (B) p_T over z frame and (C) q_T over z frame

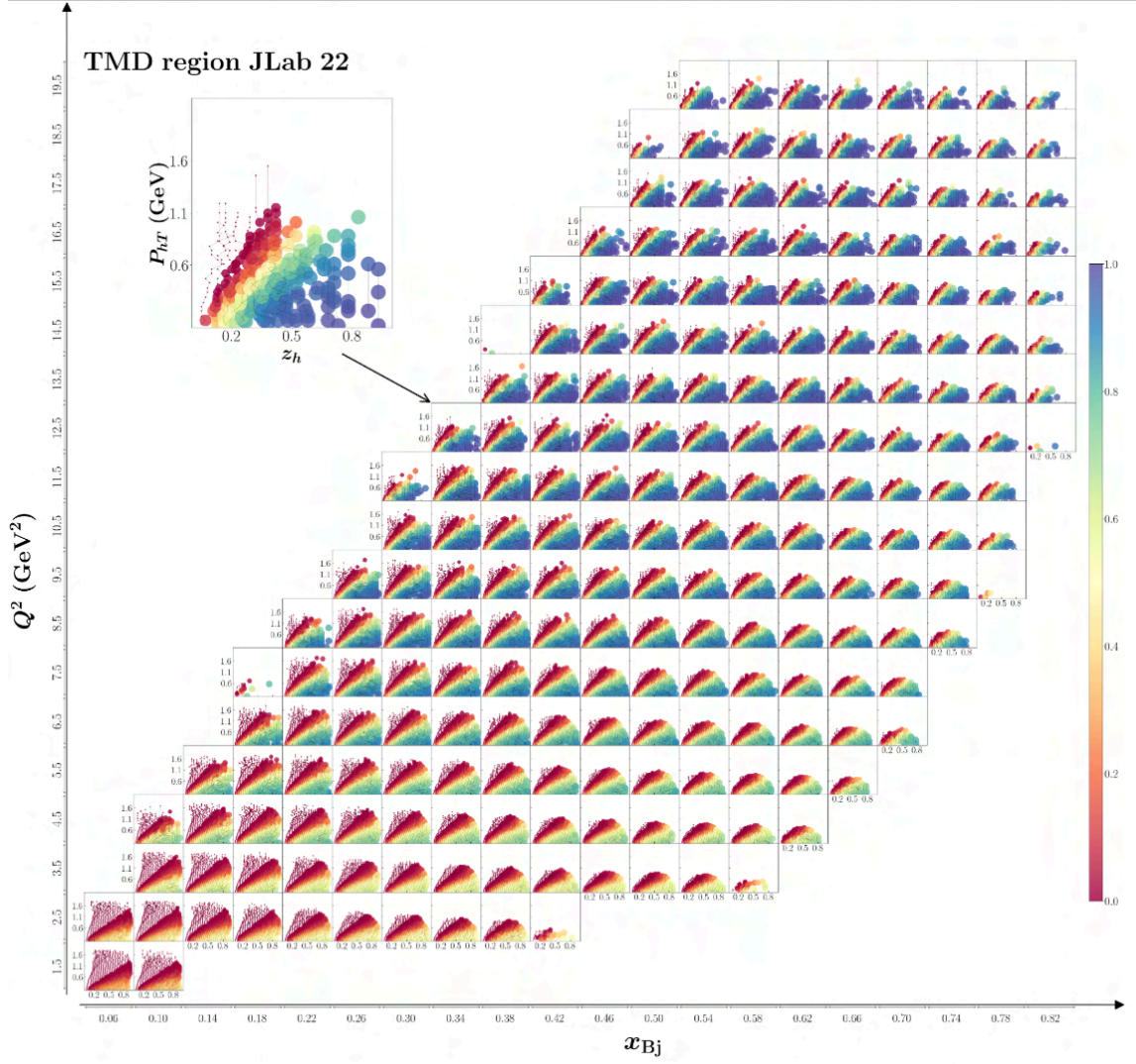


FIGURE E.2: TMD affinity results using Gaussian distributions for non-perturbative parameters

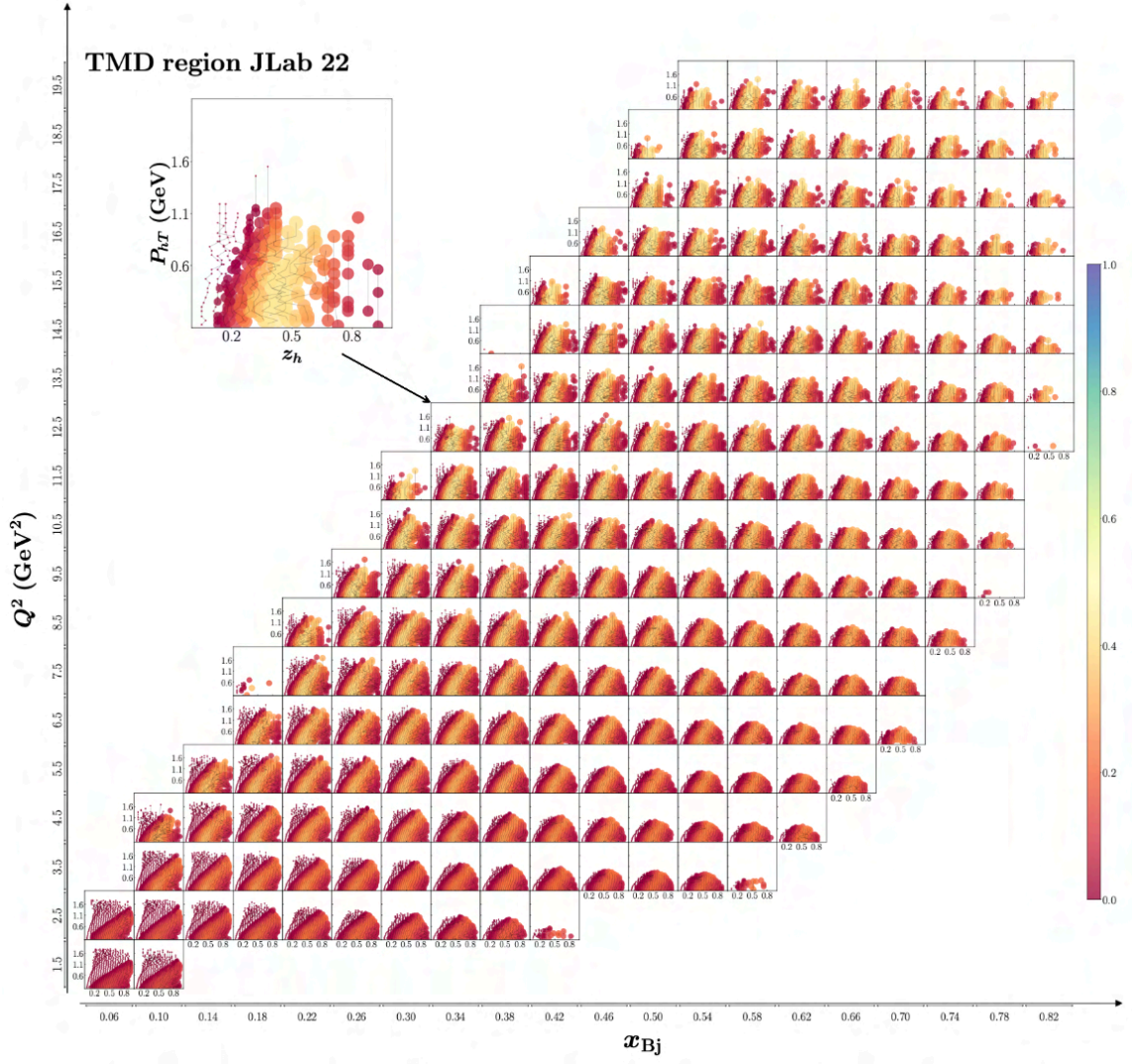


FIGURE E.3: TMD affinity results using Monte-Carlo generated distributions for non-perturbative parameters

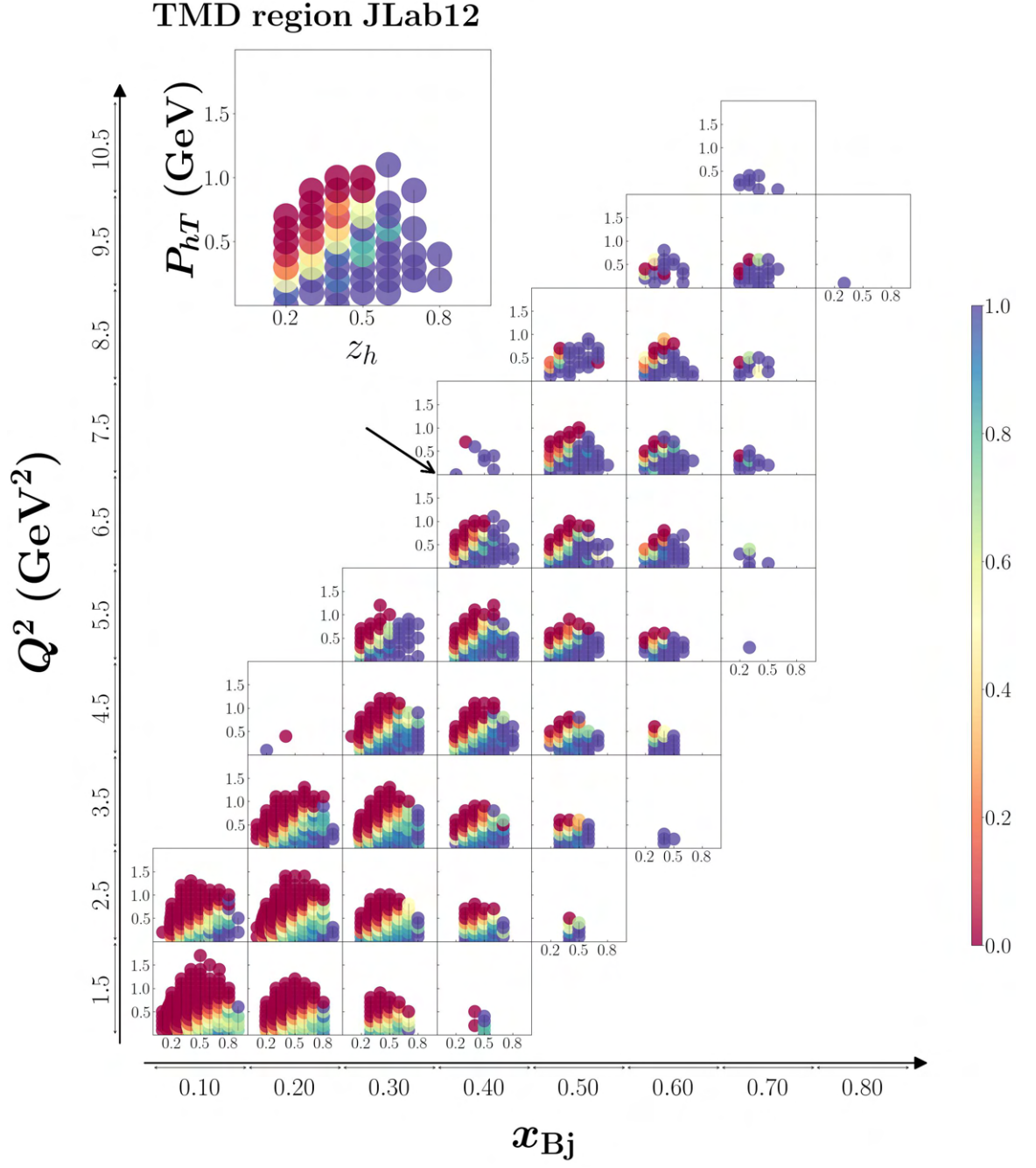


FIGURE E.4: TMD affinity results for TMD region for JLab12 dataset with π^+ outgoing hadron (JLab12 data)

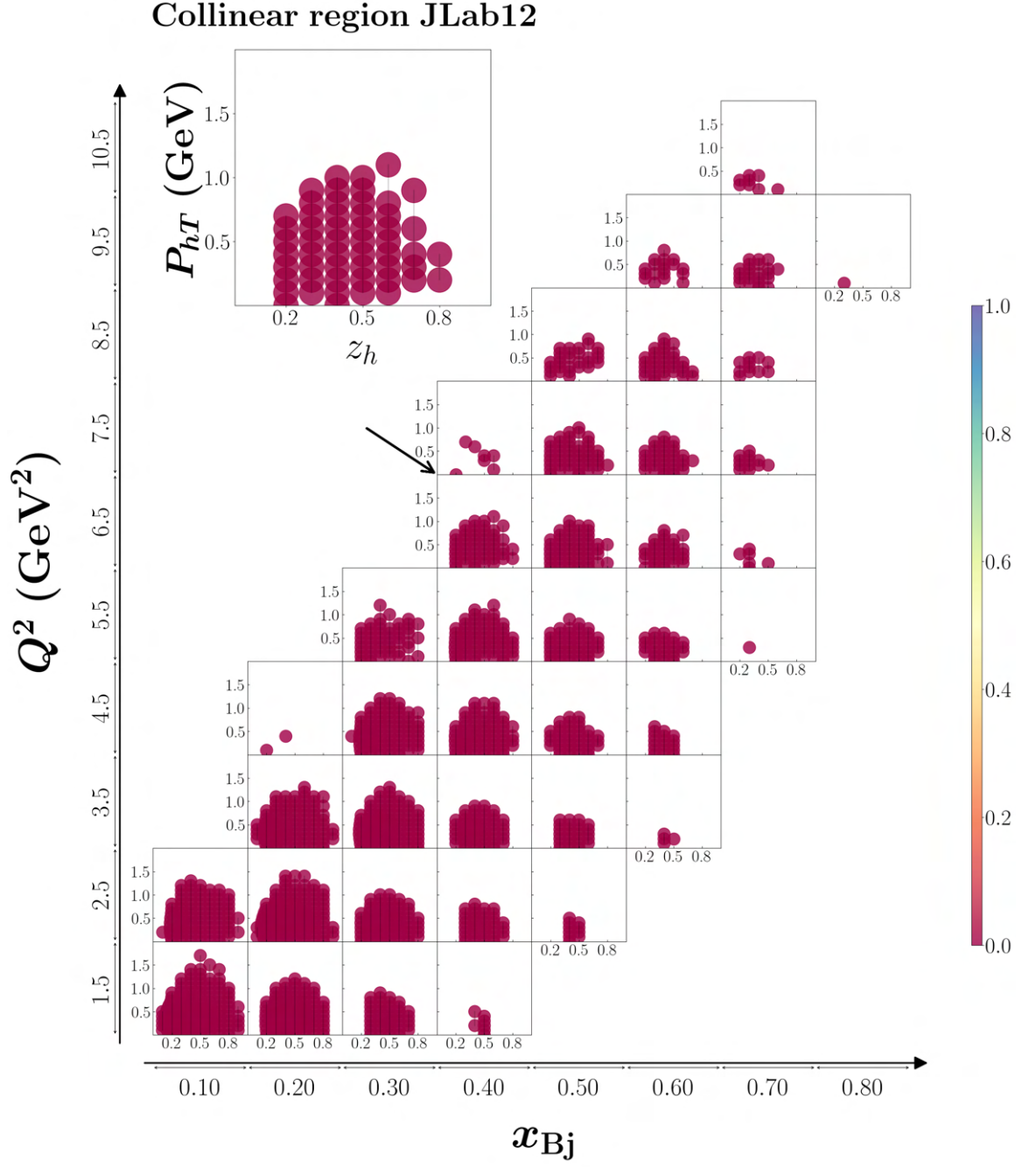


FIGURE E.5: TMD affinity results for Collinear region for JLab12 dataset with π^+ outgoing hadron (JLab12 data)

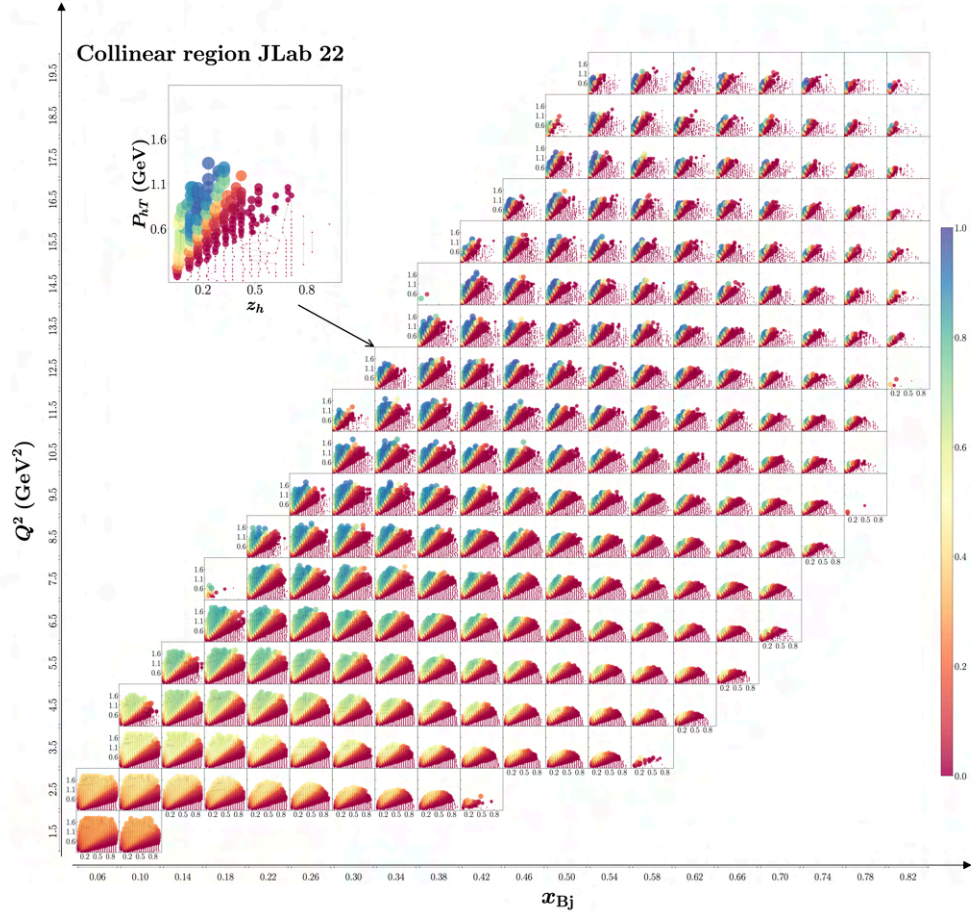


FIGURE E.6: Affinity values for Collinear region using Gaussian distributions for non-perturbative parameters (JLab22 data)

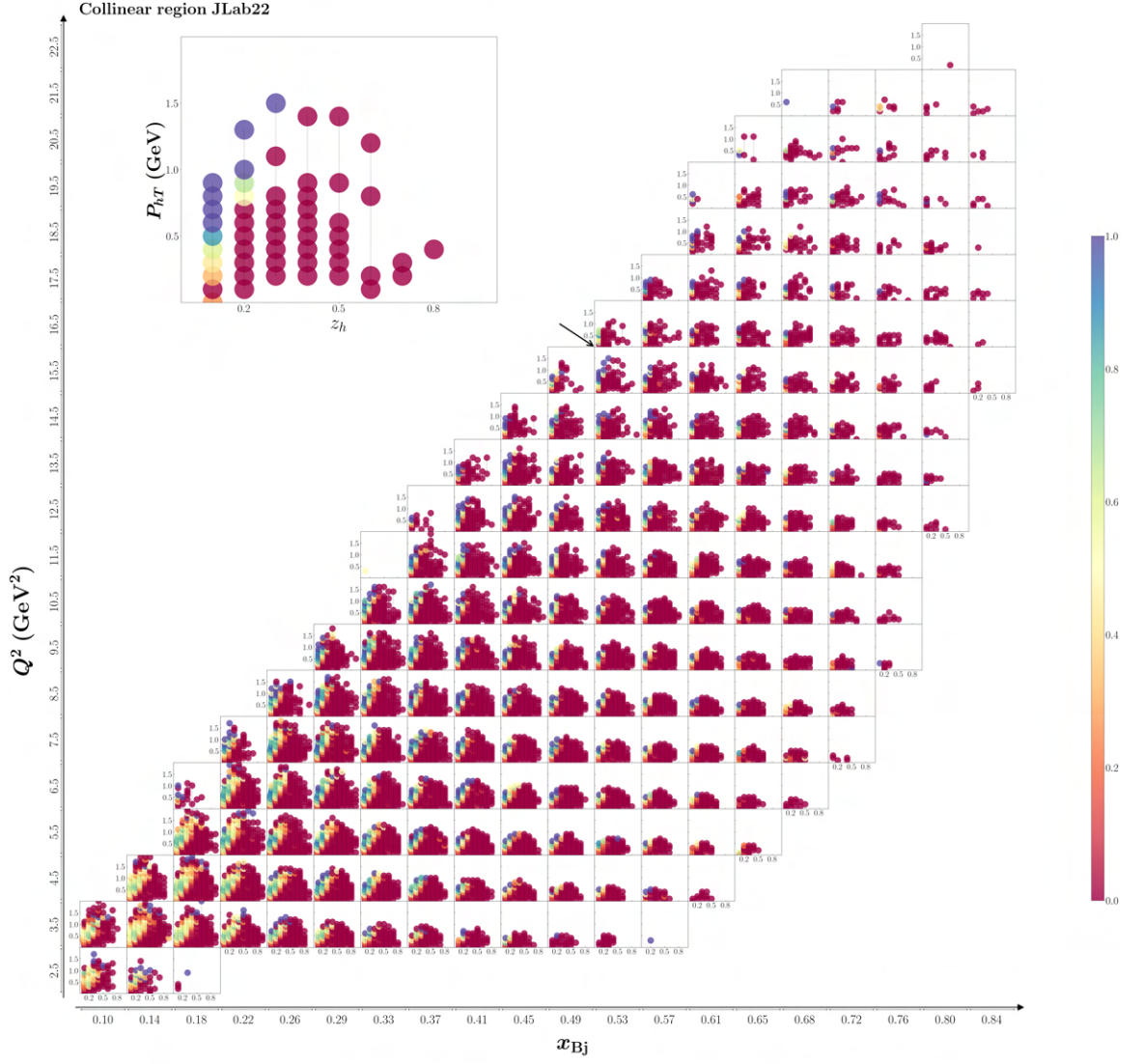


FIGURE E.7: Affinity values for Collinear region using event-by-event approach for the whole dataset (JLab22 data)

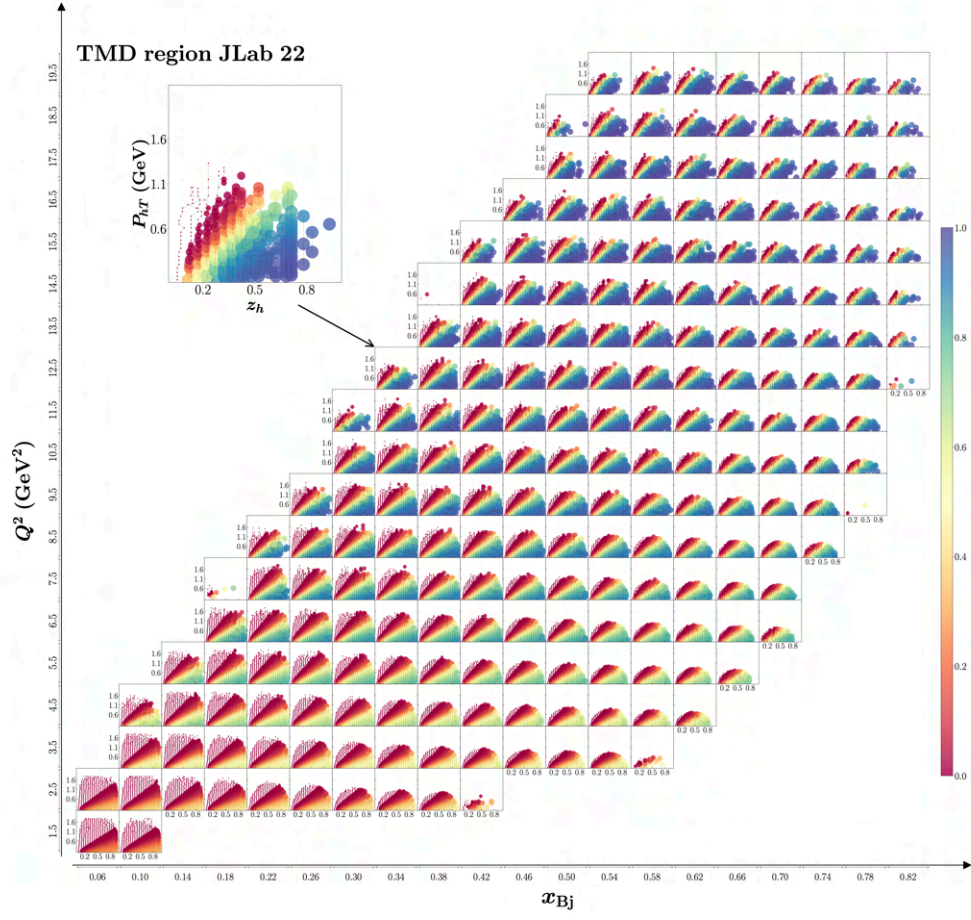


FIGURE E.8: Affinity values for TMD region using Gaussian distributions for non-perturbative parameters (JLab22 data)

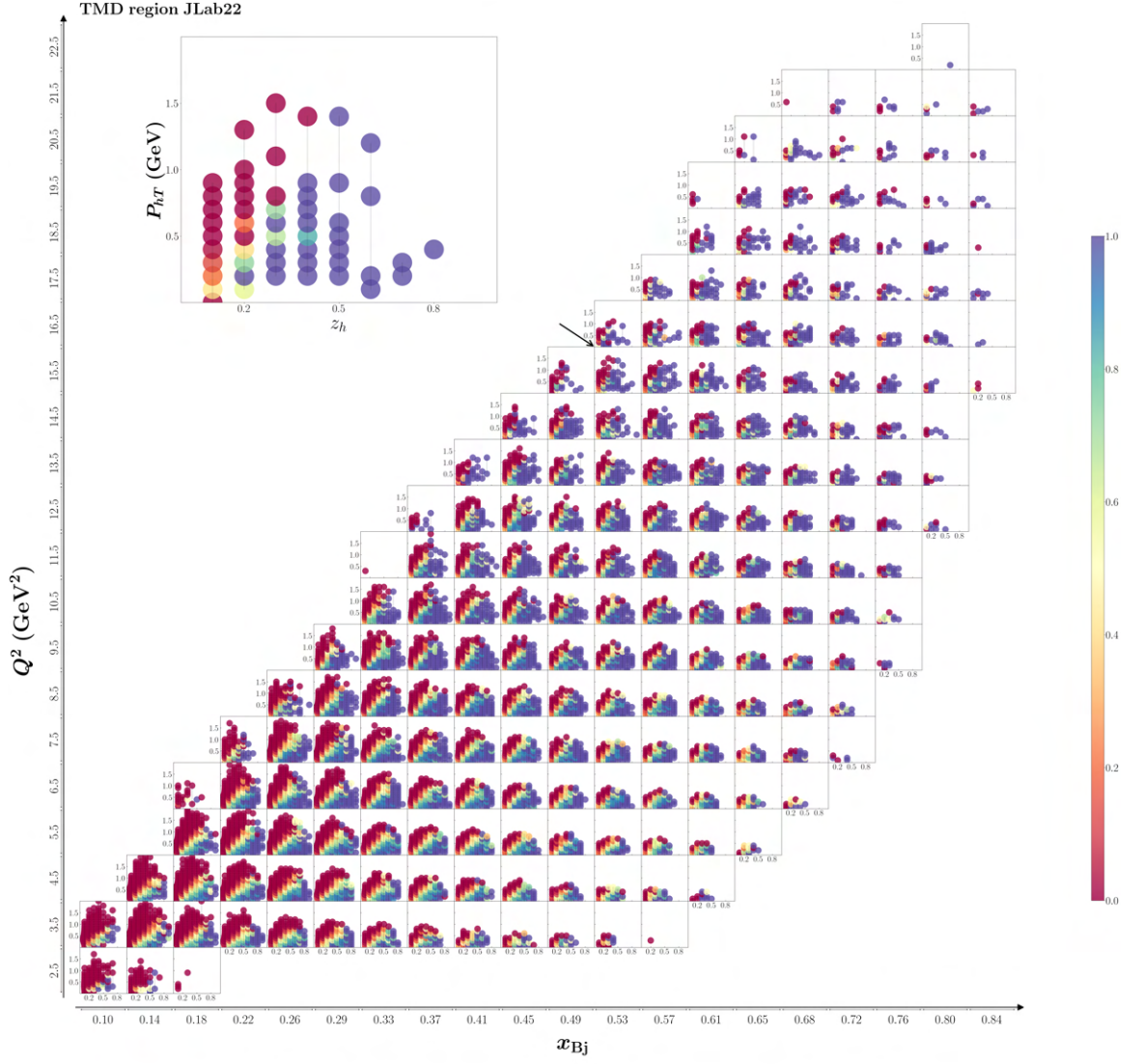


FIGURE E.9: Affinity values for TMD region using event-by-event approach for the whole dataset (JLab22 data)

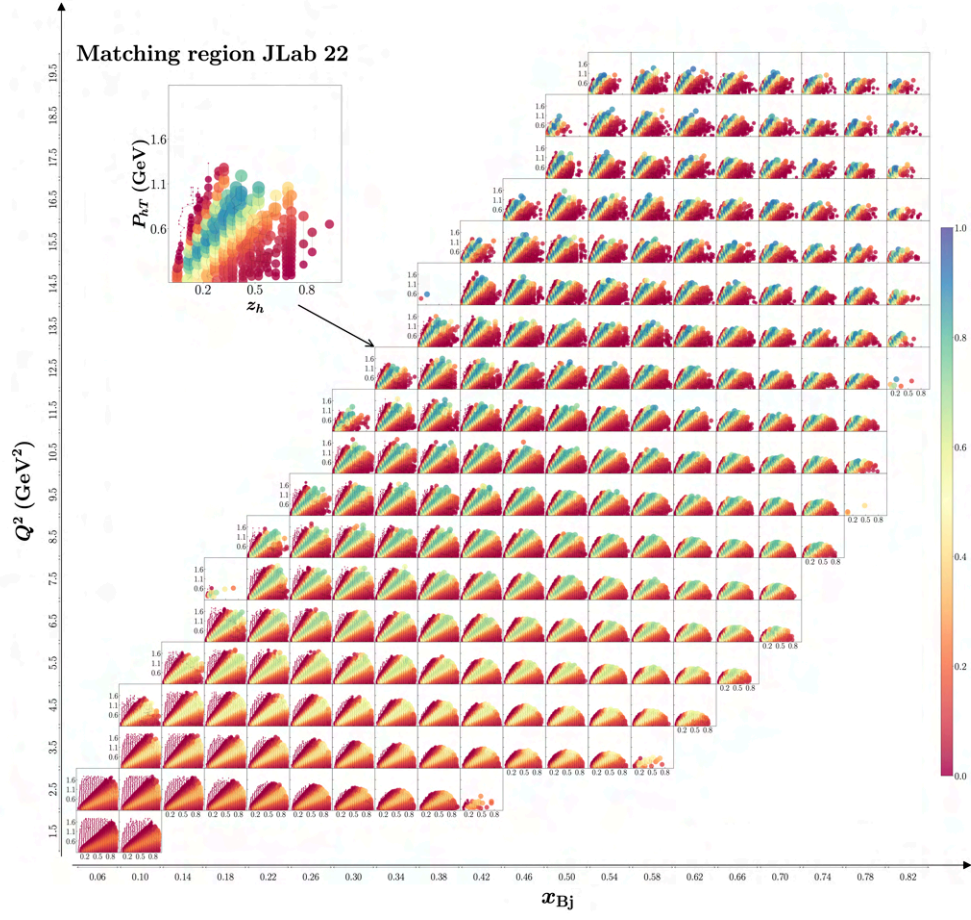


FIGURE E.10: Affinity values for TMD-Collinear Matching region using Gaussian distributions for non-perturbative parameters (JLab22 data)

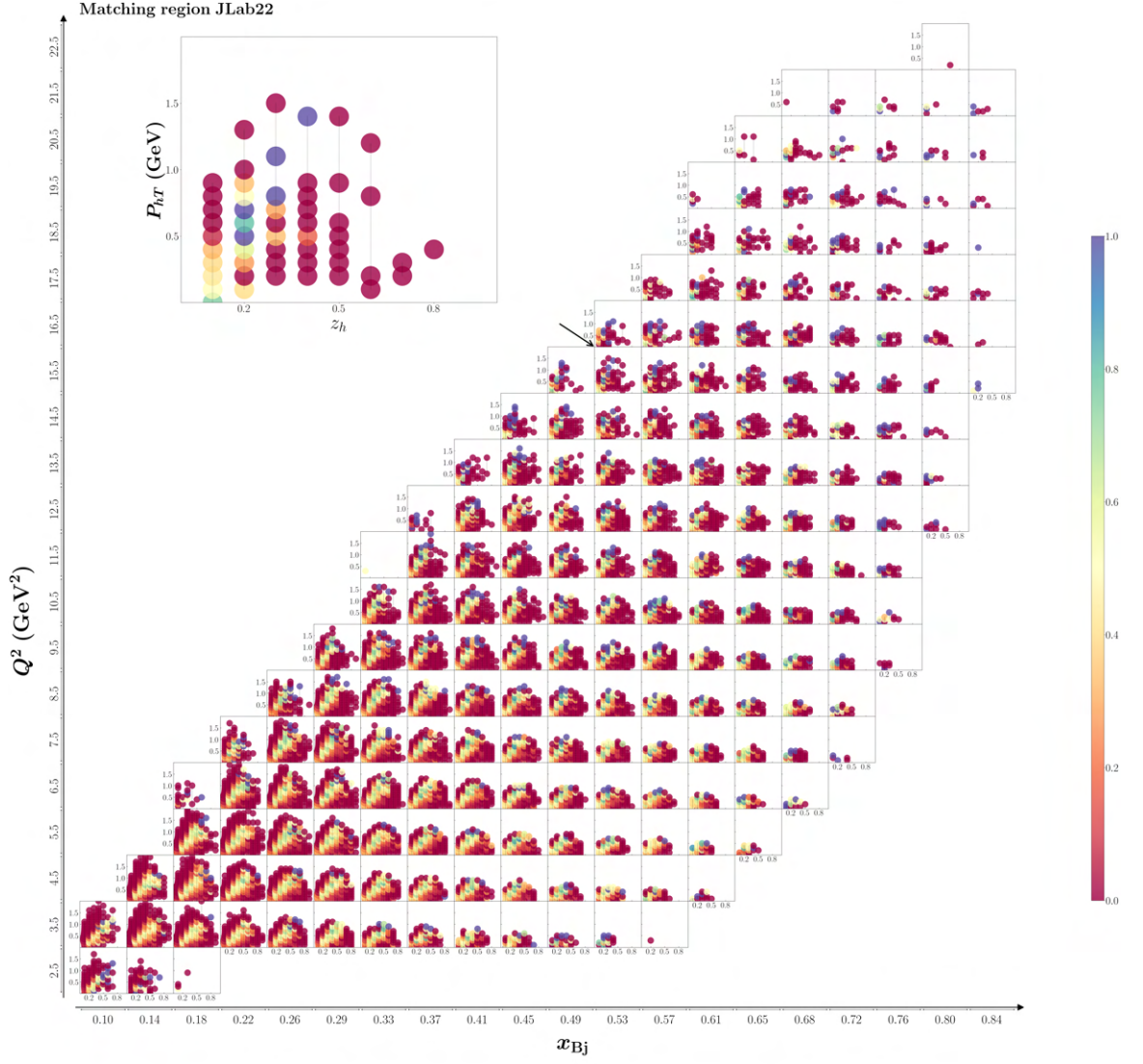


FIGURE E.11: Affinity values for TMD-Collinear Matching region using event-by-event approach for the whole dataset (JLab22 data)

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